Bulgaria Team Selection Test 2004

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Day 1

1 Let n be a positive integer. Find all positive integers m for which there exists a polynomial $f(x) = a_0 + \cdots + a_n x^n \in \mathbb{Z}[X]$ ($a_n \neq 0$) such that $\gcd(a_0, a_1, \cdots, a_n, m) = 1$ and $m|f(k)$ for each $k \in \mathbb{Z}$.

2 Find all primes $p \geq 3$ such that $p - \lfloor p/q \rfloor q$ is a square-free integer for any prime $q < p$.

3 Find the maximum possible value of the inradius of a triangle whose vertices lie in the interior, or on the boundary, of a unit square.

Day 2

1 Find the maximum possible value of the product of distinct positive integers whose sum is 2004.

2 Let H be the orthocenter of $\triangle ABC$. The points $A_1 \neq A$, $B_1 \neq B$ and $C_1 \neq C$ lie, respectively, on the circumcircles of $\triangle BCH$, $\triangle CAH$ and $\triangle ABH$ and satisfy $A_1H = B_1H = C_1H$. Denote by H_1, H_2 and H_3 the orthocenters of $\triangle A_1BC$, $\triangle B_1CA$ and $\triangle C_1AB$, respectively. Prove that $\triangle A_1B_1C_1$ and $\triangle H_1H_2H_3$ have the same orthocenter.

3 In any cell of an $n \times n$ table a number is written such that all the rows are distinct. Prove that we can remove a column such that the rows in the new table are still distinct.

Day 3

1 The points P and Q lie on the diagonals AC and BD, respectively, of a quadrilateral $ABCD$ such that $\frac{AP}{AC} + \frac{BQ}{BD} = 1$. The line PQ meets the sides AD and BC at points M and N. Prove that the circumcircles of the triangles AMP, BNQ, DMQ, and CNP are concurrent.

2 The edges of a graph with $2n$ vertices ($n \geq 4$) are colored in blue and red such that there is no blue triangle and there is no red complete subgraph with n vertices. Find the least possible number of blue edges.

3 Prove that among any $2n+1$ irrational numbers there are $n+1$ numbers such that the sum of any k of them is irrational, for all $k \in \{1, 2, 3, \ldots, n+1\}$.

Day 4

1 Find all $k > 0$ such that there exists a function $f : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions: $f(f(x,y),z) = f(x,f(y,z))$; $f(x,y) = f(y,x)$; $f(x,1) = x$; $f(zx,zy) = z^k f(x,y)$, for any $x,y,z \in [0,1]$

2 Prove that if $a,b,c \geq 1$ and $a+b+c=9$, then $\sqrt{ab+bc+ca} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$

3 A table with m rows and n columns is given. At any move one chooses some empty cells such that any two of them lie in different rows and columns, puts a white piece in any of those cells and then puts a black piece in the cells whose rows and columns contain white pieces. The game is over if it is not possible to make a move. Find the maximum possible number of white pieces that can be put on the table.