Bulgaria Team Selection Test 2007

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by bilarev

Day 1

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1 Let ABC is a triangle with $\angle BAC = \frac{\pi}{6}$ and the circumradius equal to 1. If X is a point inside or in its boundary let $m(X) = \min(AX, BX, CX)$. Find all the angles of this triangle if $\max(m(X)) = \frac{\sqrt{3}}{3}$.

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2 Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0, 1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \leq f(x) + f(y)$$

for all $x, y \in (0, 1]$.

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3 Let I be the center of the incircle of non-isosceles triangle $ABC, A_1 = AI \cap BC$ and $B_1 = BI \cap AC$. Let l_a be the line through A_1 which is parallel to AC and l_b be the line through B_1 parallel to BC. Let $l_a \cap CI = A_2$ and $l_b \cap CI = B_2$. Also $N = AA_2 \cap BB_2$ and M is the midpoint of AB. If $CN \parallel IM$ find $\frac{CN}{IM}$.

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4 Let G is a graph and x is a vertex of G. Define the transformation φ_x over G as deleting all incident edges with respect of x and drawing the edges xy such that $y \in G$ and y is not connected with x with edge in the beginning of the transformation. A graph H is called $G-$*attainable* if there exists a sequece of such transformations which transforms G in H. Let $n \in \mathbb{N}$ and $4|n$. Prove that for each graph G with $4n$ vertices and n edges there exists $G-$*attainable* graph with at least $9n^2/4$ triangles.

Day 2

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1 In isosceles triangle $ABC(AC = BC)$ the point M is in the segment AB such that $AM = 2MB$, F is the midpoint of BC and H is the orthogonal projection of M in AF. Prove that $\angle BHF = \angle ABC$.

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2 Let n, k be positive integers such that $n \geq 2k > 3$ and $A = \{1, 2, ..., n\}$. Find all n and k such that the number of k-element subsets of A is $2n - k$ times bigger than the number of 2-element subsets of A.

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3 Let $n \geq 2$ is positive integer. Find the best constant $C(n)$ such that

$$\sum_{i=1}^{n} x_i \geq C(n) \sum_{1 \leq j < i \leq n} (2x_i x_j + \sqrt{x_i x_j})$$

is true for all real numbers $x_i \in (0,1), i=1,...,n$ for which $(1-x_i)(1-x_j) \geq \frac{1}{4}, 1 \leq j < i \leq n.$

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4 Let $p=4k+3$ be a prime number. Find the number of different residues mod p of $(x^2+y^2)^2$ where $(x,p)=(y,p)=1.$
