Tests consisting of IMOSL problems are not shown.

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Test 1 28/8/2023

P1 Triangle ABC with $\angle BAC = 60^\circ$ is given. The circumcircle of ABC is Ω, and the orthocenter of ABC is H. Let S denote the midpoint of the arc BC of Ω which doesn't contain A. Point P was chosen on Ω so that $\angle HPS = 90^\circ$. Prove that there exists a circle that goes through P and S and is tangent to lines AB, AC.

P2 Let $n > 1$ be an integer. Given a simple graph G on n vertices $v_1, v_2, \dots, v_n$ we let $k(G)$ be the minimal value of k for which there exist n k-dimensional rectangular boxes $R_1, R_2, \dots, R_n$ in a k-dimensional coordinate system with edges parallel to the axes, so that for each $1 \le i < j \le n$, R_i and R_j intersect if and only if there is an edge between v_i and v_j in G.

Define M to be the maximal value of $k(G)$ over all graphs on n vertices. Calculate M as a function of n.

P3 Let n be a positive integer and p be a prime number of the form $8k+5$. A polynomial Q of degree at most 2023 and nonnegative integer coefficients less than or equal to n will be called "cool" if

$$p \mid Q(2) \cdot Q(3) \cdot \ldots \cdot Q(p-2) - 1.$$

Prove that the number of cool polynomials is even.

Test 2 6 of November, 2023

P1 Solve in positive integers:

$$x^{y^2+1} + y^{x^2+1} = 2^z$$

P2 In triangle ABC the incenter is I. The center of the excircle opposite A is I_A, and it is tangent to BC at D. The midpoint of arc BAC is N, and NI intersects (ABC) again at T. The center of (AID) is K. Prove that $TI_A \perp KI$.

P3 Let $0 < c < 1$ and n a positive integer. Alice and Bob are playing a game. Bob writes n integers on the board, not all equal. On a player's turn, they erase two numbers from the board and write their arithmetic mean instead. Alice starts and performs at most cn moves. After her, Bob makes moves until there are only two numbers left on the board. Alice wins if these two numbers are different, and otherwise, Bob wins.
For which values of c does Alice win for all large enough n?

Test 3 29/1/2024

P1 Let ABC be a triangle and let D be a point on BC so that AD bisects the angle $\angle BAC$. The common tangents of the circles (BAD), (CAD) meet at the point A'. The points B', C' are defined similarly. Show that A', B', C' are collinear.

P2 A positive integer N is given. Panda builds a tree on N vertices, and writes a real number on each vertex, so that 1 plus the number written on each vertex is greater or equal to the average of the numbers written on the neighboring vertices. Let the maximum number written be M and the minimal number written m. Mink then gives Panda $M-m$ kilograms of bamboo. What is the maximum amount of bamboo Panda can get?

P3 Find all continuous functions $f\colon \mathbb{R}_{>0}\to\mathbb{R}_{\geq 1}$ for which the following equation holds for all positive reals x, y:

$$f\left(\frac{f(x)}{y}\right)-f\left(\frac{f(y)}{x}\right)=xy\left(f(x+1)-f(y+1)\right)$$

Test 6 20/3/2024

P1 Let G be a connected (simple) graph with n vertices and at least n edges. Prove that it is possible to color the vertices of G red and blue, so that the following conditions hold:

i. There is at least one vertex of each color,
ii. There is an even number of edges connecting a red vertex to a blue vertex, and
iii. If all such edges are deleted, one is left with two connected graphs.

P2 Let n be a positive integer. Find all polynomials $Q(x)$ with integer coefficients so that the degree of $Q(x)$ is less than n and there exists an integer $m\geq 1$ for which

$$x^n-1\mid Q(x)^m-1$$

P3 Let $ABCD$ be a parallelogram. Let ω_1 be the circle passing through D tangent to AB at A. Let ω_2 be the circle passing through A tangent to CD at D. The tangents from B to ω_1 touch it at A and P. The tangents from C to ω_2 touch it at D and Q. Lines AP and DQ intersect at X. The perpendicular bisector of BC intersects AD at R.

Show that the circumcircles of triangles $\triangle PQX$, $\triangle BCR$ are concentric.

Test 8 8th of May, 2024

P1 For each positive integer n let a_n be the largest positive integer satisfying

$$(a_n)! \left| \prod_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \right.$$

Show that there are infinitely many positive integers m for which $a_{m+1} < a_m$.

P2 Triangle ABC is inscribed in circle Ω with center O. The incircle of ABC is tangent to BC, AC, AB at D, E, F respectively, and its center is I. The reflection of the tangent line to Ω at A with respect to EF will be denoted ℓ_A. We similarly define ℓ_B, ℓ_C.
Show that the orthocenter of the triangle with sides ℓ_A, ℓ_B, ℓ_C lies on OI.

P3 For a set S of at least 3 points in the plane, let d_{min} denote the minimal distance between two different points in S and d_{max} the maximal distance between two different points in S.

For a real $c > 0$, a set S will be called c-*balanced* if

$$\frac{d_{\text{max}}}{d_{\text{min}}} \leq c|S|$$

Prove that there exists a real $c > 0$ so that for every c-balanced set of points S, there exists a triangle with vertices in S that contains at least $\sqrt{|S|}$ elements of S in its interior or on its boundary.
