Bulgaria Team Selection Test 2008

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Day 1

1 Let n be a positive integer. There is a pawn in one of the cells of an $n \times n$ table. The pawn moves from an arbitrary cell of the kth column, $k \in \{1, 2, \cdots, n\}$, to an arbitrary cell in the kth row. Prove that there exists a sequence of n^2 moves such that the pawn goes through every cell of the table and finishes in the starting cell.

2 The point P lies inside, or on the boundary of, the triangle ABC. Denote by d_a, d_b and d_c the distances between P and BC, CA, and AB, respectively. Prove that $\max\{AP, BP, CP\} \ge \sqrt{d_a^2 + d_b^2 + d_c^2}$. When does the equality holds?

3 Let $\mathbb{R}^+$ be the set of positive real numbers. Find all real numbers a for which there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $3(f(x))^2 = 2f(f(x)) + ax^4$, for all $x \in \mathbb{R}^+$.

Day 2

1 For each positive integer n, denote by a_n the first digit of 2^n (base ten). Is the number $0.a_1a_2a_3\cdots$ rational?

2 In the triangle ABC, AM is median, $M \in BC$, BB_1 and CC_1 are altitudes, $C_1 \in AB$, $B_1 \in AC$. The line through A which is perpendicular to AM cuts the lines BB_1 and CC_1 at points E and F, respectively. Let k be the circumcircle of $\triangle EFM$. Suppose also that k_1 and k_2 are circles touching both EF and the arc EF of k which does not contain M. If P and Q are the points at which k_1 intersects k_2, prove that P, Q, and M are collinear.

3 Let G be a directed graph with infinitely many vertices. It is known that for each vertex the outdegree is greater than the indegree. Let O be a fixed vertex of G. For an arbitrary positive number n, let V_n be the number of vertices which can be reached from O passing through at most n edges (O counts). Find the smallest possible value of V_n.