Final Round - Costa Rica 2003

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by Jutaro

Day 1	August 26th
1	Two players A and B participate in the following game. Initially we have a pile of 2003 stones. A plays first, and he picks a divisor of 2003 and removes that number of stones from the pile. Then B picks a divisor of the number of remaining stones, and removes that number of stones from the pile, and so forth. The players who removes the last stone loses. Prove that one of the players has a winning strategy and describe it.
2	Let AB be a diameter of circle ω. ℓ is the tangent line to ω at B. Take two points C, D on ℓ such that B is between C and D. E, F are the intersections of ω and AC, AD, respectively, and G, H are the intersections of ω and CF, DE, respectively. Prove that $AH = AG$.
3	If $a > 1$ and $b > 2$ are positive integers, show that $a^b + 1 \geq b(a+1)$, and determine when equality holds.
Day 2	August 27th
4	S_1 and S_2 are two circles that intersect at distinct points P and Q. ℓ_1 and ℓ_2 are two parallel lines through P and Q. ℓ_1 intersects S_1 and S_2 at points A_1 and A_2, different from P, respectively. ℓ_2 intersects S_1 and S_2 at points B_1 and B_2, different from Q, respectively. Show that the perimeters of the triangles A_1QA_2 and B_1PB_2 are equal.
5	Each of the squares of an 8×8 board can be colored white or black. Find the number of colorings of the board such that every 2×2 square contains exactly 2 black squares and 2 white squares.
6	Say a number is *tico* if the sum of it's digits is a multiple of 2003. (i) Show that there exists a positive integer N such that the first 2003 multiples, $N, 2N, 3N, \ldots 2003N$ are all tico. (ii) Does there exist a positive integer N such that all it's multiples are tico?