Final Round - Costa Rica 2006

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Day 1

1 Consider the set $S = \{1, 2, ..., n\}$. For every $k \in S$, define $S_k = \{X \subseteq S,\ k \notin X, X \neq \emptyset\}$. Determine the value of the sum

$$S_k^* = \sum_{\{i_1, i_2, ..., i_r\} \in S_k} \frac{1}{i_1 \cdot i_2 \cdot ... \cdot i_r}$$

in fact, this problem was taken from an austrian-polish

2 If a, b, c are the sidelengths of a triangle, then prove that $\frac{3(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2} + \frac{bc+ca+ab}{a^2+b^2+c^2} \geq 2$.

3 Let ABC be a triangle. Let P, Q, R be the midpoints of BC, CA, AB respectively. Let U, V, W be the midpoints of QR, RP, PQ respectively. Let $x = AU, y = BV, z = CW$.
Prove that there exist a triangle with sides x, y, z.

Day 2

1 Let f be a function that satisfies :

$$f(x) + 2f\left(\frac{x + \frac{2001}{2}}{x - 1}\right) = 4014 - x.$$

Find $f(2004)$.

2 Let n be a positive integer, and let p be a prime, such that $n > p$.
Prove that :

$$\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod{p}.$$

3 Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

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