Final Round - Costa Rica 2008

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by trejos

1 We want to colour all the squares of an nxn board of red or black. The colorations should be such that any subsquare of $2x2$ of the board have exactly two squares of each color. If $n \geq 2$ how many such colorations are possible?

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2 Let ABC be a triangle and let P be a point on the angle bisector AD, with D on BC. Let E, F and G be the intersections of AP, BP and CP with the circumcircle of the triangle, respectively. Let H be the intersection of EF and AC, and let I be the intersection of EG and AB. Determine the geometric place of the intersection of BH and CI when P varies.

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3 Find all polinomials $P(x)$ with real coefficients, such that
$P(\sqrt{3}(a-b))+P(\sqrt{3}(b-c))+P(\sqrt{3}(c-a))=P(2a-b-c)+P(-a+2b-c)+P(-a-b+2c)$
for any a,b and c real numbers

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4 Let x, y and z be non negative reals, such that there are not two simultaneously equal to 0. Show that
$\frac{x+y}{y+z}+\frac{y+z}{x+y}+\frac{y+z}{z+x}+\frac{z+x}{y+z}+\frac{z+x}{x+y}+\frac{x+y}{z+x}\geq 5+\frac{x^2+y^2+z^2}{xy+yz+zx}$
and determine the equality cases.

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5 Let p be a prime number such that $p-1$ is a perfect square. Prove that the equation $a^2+(p-1)b^2=pc^2$
has infinite many integer solutions a, b and c with $(a,b,c)=1$

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6 Let O be the circumcircle of a ΔABC and let I be its incenter, for a point P of the plane let $f(P)$ be the point obtained by reflecting P' by the midpoint of OI, with P' the homothety of P with center O and ratio $\frac{R}{r}$ with r the inradii and R the circumradii,(understand it by $\frac{OP}{OP'}=\frac{R}{r}$). Let A_1, B_1 and C_1 the midpoints of BC, AC and AB, respectively. Show that the rays $A_1f(A)$, $B_1f(B)$ and $C_1f(C)$ concur on the incircle.
