Final Round - Costa Rica 2009

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by trejos

1 Let x and y positive real numbers such that $(1+x)(1+y)=2$. Show that $xy+\frac{1}{xy}\geq\ 6$

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2 Prove that for that for every positive integer n, the smallest integer that is greater than $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

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3 Let triangle ABC acutangle, with $m\angle ACB\leq\ m\angle ABC$. M the midpoint of side BC and P a point over the side MC. Let C_1 the circunference with center C. Let C_2 the circunference with center B. P is a point of C_1 and C_2. Let X a point on the opposite semiplane than B respecting with the straight line AP; Let Y the intersection of side XB with C_2 and Z the intersection of side XC with C_1. Let $m\angle PAX=\alpha$ and $m\angle ABC=\beta$. Find the geometric place of X if it satisfies the following conditions: $(a)\frac{XY}{XZ}=\frac{XC+CP}{XB+BP}$ $(b)\cos(\alpha)=AB\cdot\frac{\sin(\beta)}{AP}$

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4 Show that the number $3^{4^5}+4^{5^6}$ can be expresed as the product of two integers greater than 10^{2009}

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5 Suppose the polynomial $x^n+a_{n-1}x^{n-1}+...+a_1+a_0$ can be factorized as $(x+r_1)(x+r_2)...(x+r_n)$, with $r_1,r_2,...,r_n$ real numbers.
Show that $(n-1)a_{n-1}^2\geq\ 2na_{n-2}$

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6 Let ΔABC with incircle Γ, let D,E and F the tangency points of Γ with sides BC,AC and AB, respectively and let P the intersection point of AD with Γ. $a)$ Prove that BC,EF and the straight line tangent to Γ for P concur at a point A'. $b)$ Define B' and C' in an anologous way than A'. Prove that $A'-B'-C'$
