Final Round - Costa Rica 2010

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by hatchguy

1 Consider points D, E and F on sides BC, AC and AB, respectively, of a triangle ABC, such that AD, BE and CF concurr at a point G. The parallel through G to BC cuts DF and DE at H and I, respectively. Show that triangles AHG and AIG have the same areas.

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2 Consider the sequence $x_n > 0$ defined with the following recurrence relation:

$$x_1 = 0$$

and for $n > 1$

$$(n+1)^2 x_{n+1}^2 + (2^n + 4)(n+1)x_{n+1} + 2^{n+1} + 2^{2n-2} = 9n^2 x_n^2 + 36n x_n + 32.$$

Show that if n is a prime number larger or equal to 5, then x_n is an integer.

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3 Christian Reiher and Reid Barton want to open a security box, they already managed to discover the algorithm to generate the key codes and they obtained the following information:

$i)$ In the screen of the box will appear a sequence of $n+1$ numbers, $C_0 = (a_{0,1}, a_{0,2}, ..., a_{0,n+1})$

$ii)$ If the code $K = (k_1, k_2, ..., k_n)$ opens the security box then the following must happen:

a) A sequence $C_i = (a_{i,1}, a_{i,2}, ..., a_{i,n+1})$ will be asigned to each k_i defined as follows:

$a_{i,1} = 1$ and $a_{i,j} = a_{i-1,j} - k_i a_{i,j-1}$, for $i, j \geq 1$

b) The sequence (C_n) asigned to k_n satisfies that $S_n = \sum_{i=1}^{n+1} |a_i|$ has its least possible value, considering all possible sequences K.

The sequence C_0 that appears in the screen is the following:

$a_{0,1} = 1$ and a_0, i is the sum of the products of the elements of each of the subsets with $i-1$ elements of the set $A = 1, 2, 3, ..., n$, $i \geq 2$, such that $a_{0,n+1} = n!$

Find a sequence $K = (k_1, k_2, ..., k_n)$ that satisfies the conditions of the problem and show that there exists at least $n!$ of them.

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4 Find all integer solutions (a, b) of the equation

$$(a+b+3)^2 + 2ab = 3ab(a+2)(b+2)$$

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5 Let C_1 be a circle with center O and let B and C be points in C_1 such that BOC is an equilateral triangle. Let D be the midpoint of the minor arc BC of C_1. Let C_2 be the circle with center C that passes through B and O. Let E be the second intersection of C_1 and C_2. The parallel to DE through B intersects C_1 for second time in A. Let C_3 be the circumcircle of triangle AOC. The second intersection of C_2 and C_3 is F. Show that BE and BF trisect the angle $\angle ABC$.

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6 Let F be the family of all sets of positive integers with 2010 elements that satisfy the following condition:
The difference between any two of its elements is never the same as the difference of any other two of its elements. Let f be a function defined from F to the positive integers such that $f(K)$ is the biggest element of $K \in F$. Determine the least value of $f(K)$.
