Final Round - Korea 1997

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Day 1

1 A *word* is a sequence of 0 and 1 of length 8. Let x and y be two words differing in exactly three places.
Prove that the number of words differing from each of x and y in at least five places is 188.

2 The incircle of a triangle $A_1A_2A_3$ is centered at O and meets the segment OA_j at B_j , $j = 1, 2, 3$. A circle with center B_j is tangent to the two sides of the triangle having A_j as an endpoint and intersects the segment OB_j at C_j. Prove that

$$\frac{OC_1 + OC_2 + OC_3}{A_1A_2 + A_2A_3 + A_3A_1} \leq \frac{1}{4\sqrt{3}}$$

and find the conditions for equality.

3 Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

(i) if $x < y$, then $f(x) < f(y)$;

(ii) $f(xy) = g(y)f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Day 2

4 Given a positive integer n, find the number of n-digit natural numbers consisting of digits 1, 2, 3 in which any two adjacent digits are either distinct or both equal to 3.

5 For positive numbers $a_1, a_2, \dots, a_n$, we define

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad G = \sqrt[n]{a_1 \cdots a_n}, \quad H = \frac{n}{a_1^{-1} + \cdots + a_n^{-1}}$$

Prove that

(i) $\frac{A}{H} \leq -1 + 2\left(\frac{A}{G}\right)^n$, for n even

(ii) $\frac{A}{H} \leq -\frac{n-2}{n} + \frac{2(n-1)}{n}\left(\frac{A}{G}\right)^n$, for n odd

6 Let $p_1, p_2, \dots, p_r$ be distinct primes, and let $n_1, n_2, \dots, n_r$ be arbitrary natural numbers. Prove that the number of pairs of integers (x, y) such that

$$x^3 + y^3 = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

does not exceed 2^{r+1}.