Final Round - Korea 1999

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Day 1

1 We are given two triangles. Prove, that if $\angle C = \angle C'$ and $\frac{R}{r} = \frac{R'}{r'}$, then they are similar.

2 Suppose $f(x)$ is a function satisfying $|f(m+n) - f(m)| \leq \frac{n}{m}$ for all positive integers m,n. Show that for all positive integers k:

$$\sum_{i=1}^{k} \left| f(2^k) - f(2^i) \right| \leq \frac{k(k-1)}{2}$$

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3 Find all intengers n such that $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ is a divisor of $4m^2 + 1$ for some intenger m.

Day 2

1 If the equation:
$f(\frac{x-3}{x+1}) + f(\frac{3+x}{1-x}) = x$

holds true for all real x but ± 1, find $f(x)$.

2 A permutation $a_1, a_2, \cdots, a_6$ of numbers $1, 2, \cdots, 6$ can be transformed to $1, 2, \cdots, 6$ by transposing two numbers exactly four times. Find the number of such permutations.

3 Let $a_1, a_2, ..., a_{1999}$ be nonnegative real numbers satisfying the following conditions:
a. $a_1 + a_2 + ... + a_{1999} = 2$
b. $a_1 a_2 + a_2 a_3 + ... + a_{1999} a_1 = 1$.
Let $S = a_1^2 + a_2^2 + ... + a_{1999}^2$. Find the maximum and minimum values of S.