Final Round - Korea 2000

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by Sayan

Day 1

1 Prove that for any prime p, there exist integers x, y, z, and w such that $x^2 + y^2 + z^2 - wp = 0$ and $0 < w < p$

2 Determine all function f from the set of real numbers to itself such that for every x and y,

$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$

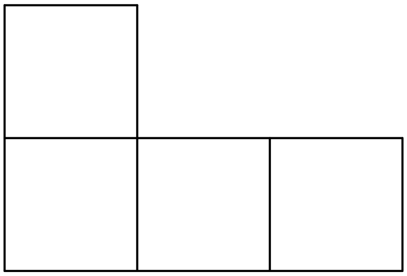
3 A rectangle $ABCD$ is inscribed in a circle with centre O. The exterior bisectors of $\angle ABD$ and $\angle ADB$ intersect at P; those of $\angle DAB$ and $\angle DBA$ intersect at Q; those of $\angle ACD$ and $\angle ADC$ intersect at R; and those of $\angle DAC$ and $\angle DCA$ intersect at S. Prove that P, Q, R, and S are concyclic.

Day 2

1 Let p be a prime such that $p \equiv 1 (\text{mod} 4)$. Evaluate

$$\sum_{k=1}^{p-1} \left(\left\lfloor \frac{2k^2}{p} \right\rfloor - 2 \left\lfloor \frac{k^2}{p} \right\rfloor \right)$$

2 Prove that an $m \times n$ rectangle can be constructed using copies of the following shape if and only if mn is a multiple of 8 where $m > 1$ and $n > 1$



3 The real numbers $a, b, c, x, y,$ and z are such that $a > b > c > 0$ and $x > y > z > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}$$

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