Final Round - Korea 2001

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Day 1

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1 Given an odd prime p, find all functions $f : Z \to Z$ satisfying the following two conditions:

(i) $f(m) = f(n)$ for all $m, n \in Z$ such that $m \equiv n \pmod{p}$;
(ii) $f(mn) = f(m)f(n)$ for all $m, n \in Z$.

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2 Let P be a given point inside a convex quadrilateral $O_1O_2O_3O_4$. For each $i = 1, 2, 3, 4$, consider the lines l that pass through P and meet the rays O_iO_{i-1} and O_iO_{i+1} (where $O_0 = O_4$ and $O_5 = O_1$) at distinct points $A_i(l)$ and $B_i(l)$, respectively. Denote $f_i(l) = PA_i(l) \cdot PB_i(l)$. Among all such lines l, let l_i be the one that minimizes f_i. Show that if $l_1 = l_3$ and $l_2 = l_4$, then the quadrilateral $O_1O_2O_3O_4$ is a parallelogram.

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3 Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be arbitrary real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 = 1$. Prove that

$$(x_1y_2 - x_2y_1)^2 \le 2\left|1 - \sum_{k=1}^{n} x_ky_k\right|$$

and find all cases of equality.

Day 2

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1 For given positive integers n and N, let P_n be the set of all polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with integer coefficients such that:

(a) $|a_j| \le N$ for $j = 0, 1, \cdots, n$;
(b) The set $\{j \mid a_j = N\}$ has at most two elements.

Find the number of elements of the set $\{f(2N) \mid f(x) \in P_n\}$.

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2 In a triangle ABC with $\angle B < 45^\circ$, D is a point on BC such that the incenter of $\triangle ABD$ coincides with the circumcenter O of $\triangle ABC$. Let P be the intersection point of the tangent lines to the circumcircle ω of $\triangle AOC$ at points A and C. The lines AD and CO meet at Q. The tangent to ω at O meets PQ at X. Prove that $XO = XD$.

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3 For a positive integer $n \geq 5$, let $a_i, b_i (i = 1, 2, \cdots, n)$ be integers satisfying the following two conditions:

(a) The pairs (a_i, b_i) are distinct for $i = 1, 2, \cdots, n$;
(b) $|a_1b_2 - a_2b_1| = |a_2b_3 - a_3b_2| = \cdots = |a_nb_1 - a_1b_n| = 1.$

Prove that there exist indices i, j such that $1 < |i - j| < n - 1$ and $|a_ib_j - a_jb_i| = 1.$