Final Round - Korea 2002

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Day 1

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1 For a prime p of the form $12k+1$ and $\mathbb{Z}_p = \{0, 1, 2, \cdots, p-1\}$, let

$$\mathbb{E}_p = \{(a,b) \mid a, b \in \mathbb{Z}_p, \quad p \nmid 4a^3 + 27b^2\}$$

For $(a,b), (a',b') \in \mathbb{E}_p$ we say that (a,b) and (a',b') are equivalent if there is a non zero element $c \in \mathbb{Z}_p$ such that $p \mid (a' - ac^4)$ and $p \mid (b' - bc^6)$. Find the maximal number of inequivalent elements in $\mathbb{E}_p$.

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2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x-y) = f(x) + xy + f(y)$ for every $x \in \mathbb{R}$ and every $y \in \{f(x) \mid x \in \mathbb{R}\}$, where $\mathbb{R}$ is the set of real numbers.

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3 The following facts are known in a mathematical contest:

(a) The number of problems tested was $n \geq 4$
(b) Each problem was solved by exactly four contestants.
(c) For each pair of problems, there is exactly one contestant who solved both problems

Assuming the number of contestants is greater than or equal to $4n$, find the minimum value of n for which there always exists a contestant who solved all the problems.

Day 2

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1 For $n \geq 3$, let $S = a_1 + a_2 + \cdots + a_n$ and $T = b_1 b_2 \cdots b_n$ for positive real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, where the numbers b_i are pairwise distinct.

(a) Find the number of distinct real zeroes of the polynomial

$$f(x) = (x-b_1)(x-b_2)\cdots(x-b_n)\sum_{j=1}^{n}\frac{a_j}{x-b_j}$$

(b) Prove the inequality

$$\frac{1}{n-1}\sum_{j=1}^{n}\left(1-\frac{a_j}{S}\right)b_j > \left(\frac{T}{S}\sum_{j=1}^{n}\frac{a_j}{b_j}\right)^{\frac{1}{n-1}}$$

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2 Let ABC be an acute triangle and let ω be its circumcircle. Let the perpendicular line from A to BC meet ω at D. Let P be a point on ω, and let Q be the foot of the perpendicular line from P to the line AB. Prove that if Q is on the outside of ω and $2\angle QPB = \angle PBC$, then D, P, Q are collinear.

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3 Let p_n be the n^{th} prime counting from the smallest prime 2 in increasing order. For example, $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$

(a) For a given $n \geq 10$, let r be the smallest integer satisfying

$$2 \leq r \leq n-2, \quad n-r+1 < p_r$$

and define $N_s = (sp_1p_2\cdots p_{r-1}) - 1$ for $s = 1, 2, \ldots, p_r$. Prove that there exists $j, 1 \leq j \leq p_r$, such that none of $p_1, p_2, \cdots, p_n$ divides N_j.

(b) Using the result of (a), find all positive integers m for which

$$p_{m+1}^2 < p_1p_2\cdots p_m$$
