Final Round - Korea 2003

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Day 1

1 Some computers of a computer room have a following network. Each computers are connected by three cable to three computers. Two arbitrary computers can exchange data directly or indirectly (through other computers). Now let's remove K computers so that there are two computers, which can not exchange data, or there is one computer left. Let k be the minimum value of K. Let's remove L cable from original network so that there are two computers, which can not exchange data. Let l be the minimum value of L. Show that $k = l$.

2 Let M be the intersection of two diagonal, AC and BD, of a rhombus $ABCD$, where angle $A < 90^\circ$. Construct O on segment MC so that $OB < OC$ and let $t = \frac{MA}{MO}$, provided that $O \neq M$. Construct a circle that has O as centre and goes through B and D. Let the intersections between the circle and AB be B and X. Let the intersections between the circle and BC be B and Y. Let the intersections of AC with DX and DY be P and Q, respectively. Express $\frac{OQ}{OP}$ in terms of t.

3 Show that the equation, $2x^4 + 2x^2y^2 + y^4 = z^2$, does not have integer solution when $x \neq 0$.

Day 2

1 Let P, Q, and R be the points where the incircle of a triangle ABC touches the sides AB, BC, and CA, respectively.
Prove the inequality $\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6$.

2 For a positive integer, m, answer the following questions.

1) Show that $2^{m+1} + 1$ is a prime number, when $2^{m+1} + 1$ is a factor of $3^{2^m} + 1$.

2) Is converse of 1) true?

3 There are n distinct points on a circumference. Choose one of the points. Connect this point and the mth point from the chosen point counterclockwise with a segment. Connect this mth point and the mth point from this mth point counterclockwise with a segment. Repeat such steps until no new segment is constructed. From the intersections of the segments, let the number of the intersections - which are in the circle - be I. Answer the following questions (m and n are positive integers that are relatively prime and they satisfy $6 \leq 2m < n$).

1) When the n points take different positions, express the maximum value of I in terms of m

and n.

2) Prove that $I \geq n$. Prove that there is a case, which is $I = n$, when $m = 3$ and n is arbitrary even number that satisfies the condition.

