Final Round - Korea 2004

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Day 1

1 An isosceles triangle with $AB = AC$ has an inscribed circle O, which touches its sides BC, CA, AB at K, L, M respectively. The lines OL and KM intersect at N; the lines BN and CA intersect at Q. Let P be the foot of the perpendicular from A on BQ. Suppose that $BP = AP + 2 \cdot PQ$. Then, what values can the ratio $\frac{AB}{BC}$ assume?

2 Prove that the equation $3y^2 = x^4 + x$ has no positive integer solutions.

3 2004 computers make up a network using several cables. If for a subset S in the set of all computers, there isn't a cable that connects two computers in S, S is called independant. One lets the arbitrary independant set consists at most 50 computers, and uses the least number of cables.

(1) Let $c(L)$ be the number of cables which connects the computer L. Prove that for two computers A, B, $c(A) = c(B)$ if there is a cable which connects A and B, $|c(A) - c(B)| \leq 1$ otherwise.

(2) Determine the number of used cables.

Day 2

1 On a circle there are n points such that every point has a distinct number. Determine the number of ways of choosing k points such that for any point there are at least 3 points between this point and the nearest point. (clockwise) $(n, k \geq 2)$

2 An acute triangle ABC has circumradius R, inradius r. A is the biggest angle among A, B, C. Let M be the midpoint of BC, and X be the intersection of two lines that touches circumcircle of ABC and goes through B, C respectively. Prove the following inequality : $\frac{r}{R} \geq \frac{AM}{AX}$.

3 For prime number p, let $f_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$.

(1) When p divides m, prove that there exists a prime number that is coprime with $m(m-1)$ and divides $f_p(m)$.

(2) Prove that there are infinitely many positive integers n such that $pn + 1$ is prime number.