Final Round - Korea 2005

www.artofproblemsolving.com/community/c3541

by N.T.TUAN, Rushil

Day 1

1 Find all natural numbers that can be expressed in a unique way as a sum of ve or less perfect squares.

2 Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers and let α_n be the arithmetic mean of $a_1, ..., a_n$. Prove that for all positive integers N ,

$$\sum_{n=1}^{N} \alpha_n^2 \leq 4 \sum_{n=1}^{N} a_n^2.$$

3 In a trapezoid $ABCD$ with $AD \parallel BC, O_1, O_2, O_3, O_4$ denote the circles with diameters AB, BC, CD, DA, respectively. Show that there exists a circle with center inside the trapezoid which is tangent to all the four circles $O_1, ..., O_4$ if and only if $ABCD$ is a parallelogram.

Day 2

4 In the following, the point of intersection of two lines g and h will be abbreviated as $g \cap h$.

Suppose ABC is a triangle in which $\angle A = 90^\circ$ and $\angle B > \angle C$. Let O be the circumcircle of the triangle ABC. Let l_A and l_B be the tangents to the circle O at A and B, respectively.

Let $BC \cap l_A = S$ and $AC \cap l_B = D$. Furthermore, let $AB \cap DS = E$, and let $CE \cap l_A = T$. Denote by P the foot of the perpendicular from E on l_A. Denote by Q the point of intersection of the line CP with the circle O (different from C). Denote by R be the point of intersection of the line QT with the circle O (different from Q). Finally, define $U = BR \cap l_A$. Prove that

$$\frac{SU \cdot SP}{TU \cdot TP} = \frac{SA^2}{TA^2}.$$

5 Find all positive integers m and n such that both $3^m + 1$ and $3^n + 1$ are divisible by mn.

6 A set P consists of 2005 distinct prime numbers. Let A be the set of all possible products of 1002 elements of P , and B be the set of all products of 1003 elements of P . Find a one-to-one correspondance f from A to B with the property that a divides $f(a)$ for all $a \in A$.