Final Round - Korea 2006

www.artofproblemsolving.com/community/c3542

by N.T.TUAN

Day 1

1 In a triangle ABC with $AB \neq AC$, the incircle touches the sides BC, CA, AB at D, E, F , respectively. Line AD meets the incircle again at P . The line EF and the line through P perpendicular to AD meet at Q. Line AQ intersects DE at X and DF at Y . Prove that A is the midpoint of XY.

2 For a positive integer a, let S_a be the set of primes p for which there exists an odd integer b such that p divides $(2^{2^a})^b - 1$. Prove that for every a there exist innitely many primes that are not contained in S_a.

3 Three schools A, B and C , each with ve players denoted a_i, b_i, c_i respectively, take part in a chess tournament. The tournament is held following the rules:
(i) Players from each school have matches in order with respect to indices, and defeated players are eliminated; the rst match is between a_1 and b_1.
(ii) If $y_j \in Y$ defeats $x_i \in X$, his next opponent should be from the remaining school if not all of its players are eliminated; otherwise his next oponent is x_{i+1} . The tournament is over when two schools are completely eliminated.
(iii) When x_i wins a match, its school wins 10^{i-1} points.
At the end of the tournament, schools A, B, C scored P_A, P_B, P_C respectively. Find the remainder of the number of possible triples (P_A, P_B, P_C) upon division by 8.

Day 2

1 Given three distinct real numbers a_1, a_2, a_3 , dene $b_j = (1 + \frac{a_j a_i}{a_j - a_i})(1 + \frac{a_j a_k}{a_j - a_k})$, where $\{i, j, k\} = \{1, 2, 3\}$.
Prove that $1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq (1 + |a_1|)(1 + |a_2|)(1 + |a_3|)$ and nd the cases of equality.

2 In a convex hexagon $ABCDEF$ triangles ABC, CDE, EFA are similar. Find conditions on these triangles under which triangle ACE is equilateral if and only if so is BDF.

3 A positive integer N is said to be $n-$ good if
(i) N has at least n distinct prime divisors, and
(ii) there exist distinct positive divisors $1, x_2, ..., x_n$ whose sum is N .
Show that there exists an $n-$ good number for each $n \geq 6$.