Final Round - Korea 2007

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1 Let O be the circumcenter of an acute triangle ABC and let k be the circle with center P that is tangent to O at A and tangent to side BC at D. Circle k meets AB and AC again at E and F respectively. The lines OP and EP meet k again at I and G. Lines BO and IG intersect at H. Prove that $\frac{DF^2}{AF} = GH$.

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2 Given a 4×4 squares table. How many ways that we can fill the table with $\{0, 1\}$ such that two neighbor squares (have one common side) have product which is equal to 0?

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3 Find all triples of (x, y, z) of positive intergers satisfying $1 + 4^x + 4^y = z^2$.

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4 Find all pairs (p, q) of primes such that $p^p + q^q + 1$ is divisible by pq.

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5 For the vertex A of a triangle ABC, let l_a be the distance between the projections on AB and AC of the intersection of the angle bisector of $\angle A$ with side BC. Define l_b and l_c analogously. If l is the perimeter of triangle ABC, prove that

$$\frac{l_a l_b l_c}{l^3} \le \frac{1}{64}.$$

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6 Let f:N→N be a function satisfying $kf(n) \le f(kn) \le kf(n) + k - 1$ for all $k, n \in N$.

(a)Prove that $f(a) + f(b) \le f(a + b) \le f(a) + f(b) + 1$ for all $a, b \in N$.

(b)If f satisfies $f(2007n) \le 2007f(n) + 200$ for every $n \in N$, show that there exists $c \in N$ such that $f(2007c) = 2007f(c)$.
