Final Round - Korea 2010

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Day 1 March 27th

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1 Given an arbitrary triangle ABC, denote by P, Q, R the intersections of the incircle with sides BC, CA, AB respectively. Let the area of triangle ABC be T, and its perimeter L. Prove that the inequality

$$\left(\frac{AB}{PQ}\right)^3+\left(\frac{BC}{QR}\right)^3+\left(\frac{CA}{RP}\right)^3 \geq \frac{2}{\sqrt{3}}\cdot\frac{L^2}{T}$$

holds.

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2 Let I be the incentre and O the circumcentre of a given acute triangle ABC. The incircle is tangent to BC at D. Assume that $\angle B < \angle C$ and the segments AO and HD are parallel, where H is the orthocentre of triangle ABC. Let the intersection of the line OD and AH be E. If the midpoint of CI is F, prove that E, F, I, O are concyclic.

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3 There are n websites $1, 2, \dots, n$ ($n \geq 2$). If there is a link from website i to j, we can use this link so we can move website i to j.
For all $i \in \{1, 2, \dots, n-1\}$, there is a link from website i to $i+1$.
Prove that we can add less or equal than $3(n-1)\log_2(\log_2 n)$ links so that for all integers $1 \leq i < j \leq n$, starting with website i, and using at most three links to website j. (If we use a link, website's number should increase. For example, No.7 to 4 is impossible).

Sorry for my bad English.

Day 2 March 28th

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4 Given is a trapezoid $ABCD$ where AB and CD are parallel, and A, B, C, D are clockwise in this order. Let Γ_1 be the circle with center A passing through B, Γ_2 be the circle with center C passing through D. The intersection of line BD and Γ_1 is P $(\neq B, D)$. Denote by Γ the circle with diameter PD, and let Γ and Γ_1 meet at $X(\neq P)$. Γ and Γ_2 meet at Y. If the circumcircle of triangle XBY and Γ_2 meet at Q, prove that B, D, Q are collinear.

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5 On a circular table are sitting $2n$ people, equally spaced in between. m cookies are given to these people, and they give cookies to their neighbors according to the following rule.

(i) One may give cookies only to people adjacent to himself.
(ii) In order to give a cookie to one's neighbor, one must eat a cookie.

Select arbitrarily a person A sitting on the table. Find the minimum value m such that there is a strategy in which A can eventually receive a cookie, independent of the distribution of cookies at the beginning.

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6 An arbitrary prime p is given. If an integer sequence $(n_1, n_2, \cdots, n_k)$ satisfying the conditions
- For all $i = 1, 2, \cdots, k$, $n_i \geq \frac{p+1}{2}$
- For all $i = 1, 2, \cdots, k$, $p^{n_i} - 1$ is divisible by n_{i+1}, and $\frac{p^{n_i}-1}{n_{i+1}}$ is coprime to n_{i+1}. Let $n_{k+1} = n_1$.
exists not for $k = 1$, but exists for some $k \geq 2$, then call the prime a good prime.
Prove that a prime is good iff it is not 2.
