Final Round - Korea 2011

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Day 1

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1 Prove that there is no positive integers x, y, z satisfying

$$x^2y^4 - x^4y^2 + 4x^2y^2z^2 + x^2z^4 - y^2z^4 = 0$$

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2 ABC is an acute triangle. P(different from B, C) is a point on side BC. H is an orthocenter, and D is a foot of perpendicular from H to AP.
The circumcircle of the triangle ABD and ACD is O_1 and O_2, respectively.
A line l parallel to BC passes D and meet O_1 and O_2 again at X and Y, respectively. l meets AB at E, and AC at F. Two lines XB and YC intersect at Z.
Prove that $ZE = ZF$ is a necessary and sufficient condition for $BP = CP$.

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3 There are n boys $a_1, a_2, \ldots, a_n$ and n girls $b_1, b_2, \ldots, b_n$. Some pairs of them are connected. Any two boys or two girls are not connected, and a_i and b_i are not connected for all $i \in \{1, 2, \ldots, n\}$.
Now all boys and girls are divided into several groups satisfying two conditions:
(i) Every groups contains an equal number of boys and girls.
(ii) There is no connected pair in the same group.
Assume that the number of connected pairs is m. Show that we can make the number of groups not larger than $\max\left\{2, \dfrac{2m}{n} + 1\right\}$.

Day 2

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1 Find the maximal value of the following expression, if a, b, c are nonnegative and $a + b + c = 1$.

$$\frac{1}{a^2 - 4a + 9} + \frac{1}{b^2 - 4b + 9} + \frac{1}{c^2 - 4c + 9}$$

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2 ABC is a triangle such that $AC < AB < BC$ and D is a point on side AB satisfying $AC = AD$. The circumcircle of ABC meets with the bisector of angle A again at E and meets with CD again at F. K is an intersection point of BC and DE. Prove that $CK = AC$ is a necessary and sufficient condition for $DK \cdot EF = AC \cdot DF$.

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3 There is a chessboard with m columns and n rows. In each blanks, an integer is given. If a rectangle R (in this chessboard) has an integer h satisfying the following two conditions, we

call R as a 'shelf'.
(i) All integers contained in R are bigger than h.
(ii) All integers in blanks, which are not contained in R but meet with R at a vertex or a side, are not bigger than h.
Assume that all integers are given to make shelves as much as possible. Find the number of shelves.