Final Round - Korea 2012

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Day 1

1 Let x, y, z be positive real numbers. Prove that

$$\frac{2x^2+xy}{(y+\sqrt{zx}+z)^2}+\frac{2y^2+yz}{(z+\sqrt{xy}+x)^2}+\frac{2z^2+zx}{(x+\sqrt{yz}+y)^2}\geq 1$$

2 For a triangle ABC which $\angle B \neq 90^\circ$ and $AB \neq AC$, define P_{ABC} as follows ;

Let I be the incenter of triangle ABC, and let D, E, F be the intersection points with the incircle and segments BC, CA, AB. Two lines AB and DI meet at S and let T be the intersection point of line DE and the line which is perpendicular with DF at F. The line ST intersects line EF at R. Now define P_{ABC} be one of the intersection points of the incircle and the circle with diameter IR, which is located in other side with A about IR.

Now think of an isosceles triangle XYZ such that $XZ = YZ > XY$. Let W be the point on the side YZ such that $WY < XY$ and Let $K = P_{YXW}$ and $L = P_{ZXW}$. Prove that $2KL \leq XY$.

3 $A_1, A_2, \cdots, A_n$ are given subsets. Let $S = \{1, 2, \cdots, n\}$. For any $X \subset S$, let

$$N(X) = \{i \in S - X \mid \forall j \in X,\ A_i \cap A_j \neq \emptyset\}$$

Let m be an integer such that $3 \leq m \leq n-2$. Prove that there exist $X \subset S$ such that $|X| = m$ and $|N(X)| \neq 1$.

Day 2

1 Let ABC be an acute triangle. Let H be the foot of perpendicular from A to BC. D, E are the points on AB, AC and let F, G be the foot of perpendicular from D, E to BC. Assume that $DG \cap EF$ is on AH. Let P be the foot of perpendicular from E to DH. Prove that $\angle APE = \angle CPE$.

2 Let n be a given positive integer. Prove that there exist infinitely many integer triples (x, y, z) such that

$$nx^2 + y^3 = z^4,\ \gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1.$$

3 Let M be the set of positive integers which do not have a prime divisor greater than 3. For any infinite family of subsets of M, say $A_1, A_2, \ldots$, prove that there exist $i \neq j$ such that for each $x \in A_i$ there exists some $y \in A_j$ such that $y \mid x$.