Final Round - Korea 2013

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1 For a triangle $\triangle ABC(\angle B > \angle C)$, D is a point on AC satisfying $\angle ABD = \angle C$. Let I be the incenter of $\triangle ABC$, and circumcircle of $\triangle CDI$ meets AI at $E(\neq I)$. The line passing E and parallel to AB meets the line BD at P. Let J be the incenter of $\triangle ABD$, and A' be the point such that $AI = IA'$. Let Q be the intersection point of JP and $A'C$. Prove that $QJ = QA'$.

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2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying following conditions.
(a) $f(x) \geq 0$ for all $x \in \mathbb{R}$.
(b) For $a, b, c, d \in \mathbb{R}$ with $ab + bc + cd = 0$, equality $f(a-b) + f(c-d) = f(a) + f(b+c) + f(d)$ holds.

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3 For a positive integer $n \geq 2$, define set $T = \{(i,j) | 1 \leq i < j \leq n, i|j\}$. For nonnegative real numbers $x_1, x_2, \cdots, x_n$ with $x_1 + x_2 + \cdots + x_n = 1$, find the maximum value of

$$\sum_{(i,j)\in T} x_i x_j$$

in terms of n.

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4 For a triangle ABC, let B_1, C_1 be the excenters of B, C. Line B_1C_1 meets with the circumcircle of $\triangle ABC$ at point $D(\neq A)$. E is the point which satisfies $B_1E \perp CA$ and $C_1E \perp AB$. Let w be the circumcircle of $\triangle ADE$. The tangent to the circle w at D meets AE at F. G, H are the points on AE, w such that $DGH \perp AE$. The circumcircle of $\triangle HGF$ meets w at point $I(\neq H)$, and J be the foot of perpendicular from D to AH. Prove that AI passes the midpoint of DJ.

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5 Two coprime positive integers a, b are given. Integer sequence $\{a_n\}, \{b_n\}$ satisties

$$(a + b\sqrt{2})^{2n} = a_n + b_n\sqrt{2}$$

Find all prime numbers p such that there exist positive integer $n \leq p$ satisfying $p | b_n$.

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6 For any permutation $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$, and define

$$A = \{i | i > f(i)\}$$

$$B = \{(i,j) | i < j \leq f(j) < f(i) \; or \; f(j) < f(i) < i < j\}$$

$$C = \{(i,j) | i < j \leq f(i) < f(j) \; or \; f(i) < f(j) < i < j\}$$

$$D = \{(i,j) | i < j \; and \; f(i) > f(j)\}$$

Prove that $|A| + 2|B| + |C| = |D|$.
