Final Round - Korea 2014

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by syk0526

Day 1 March 22nd

1 Suppose x, y, z are positive numbers such that $x+y+z=1$. Prove that

$$\frac{(1+xy+yz+zx)(1+3x^3+3y^3+3z^3)}{9(x+y)(y+z)(z+x)} \geq \left(\frac{x\sqrt{1+x}}{\sqrt[4]{3+9x^2}}+\frac{y\sqrt{1+y}}{\sqrt[4]{3+9y^2}}+\frac{z\sqrt{1+z}}{\sqrt[4]{3+9z^2}}\right)^2.$$

2 Let ABC be a isosceles triangle with $AC=BC>AB$. Let E,F be the midpoints of segments AC,AB, and let l be the perpendicular bisector of AC. Let l meets AB at K, the line through B parallel to KC meets AC at point L, and line FL meets l at W. Let P be a point on segment BF. Let H be the orthocenter of triangle ACP and line BH and CP meet at point J. Line FJ meets l at M. Prove that $AW=PW$ if and only if B lies on the circumcircle of EFM.

3 There are n students sitting on a round table. You collect all of n name tags and give them back arbitrarily. Each student gets one of n name tags. Now n students repeat following operation: The students who have their own name tags exit the table. The other students give their name tags to the student who is sitting right to him.

Find the number of ways giving name tags such that there exist a student who don't exit the table after 4 operations.

Day 2 March 23rd

4 Let ABC be a isosceles triangle with $AC=BC$. Let D a point on a line BA such that A lies between B,D. Let O_1 be the circumcircle of triangle DAC. O_1 meets BC at point E. Let F be the point on BC such that FD is tangent to circle O_1, and let O_2 be the circumcircle of DBF. Two circles O_1,O_2 meet at point $G(\neq D)$. Let O be the circumcenter of triangle BEG. Prove that the line FG is tangent to circle O if and only if $DG \perp FO$.

5 Let $p>5$ be a prime. Suppose that there exist integer k such that k^2+5 is divisible by p. Prove that there exist two positive integers m,n satisfying $p^2=m^2+5n^2$.

6 In an island there are n castles, and each castle is in country A or B. There is one commander per castle, and each commander belongs to the same country as the castle he's initially in. There are some (two-way) roads between castles (there may be roads between castles of different countries), and call two castles adjacent if there is a road between them.

Prove that the following two statements are equivalent:

(1) If some commanders from country B move to attack an adjacent castle in country A, some commanders from country A could appropriately move in defense to adjacent castles in country A so that in every castle of country A, the number of country A's commanders defending that castle is not less than the number of country B's commanders attacking that castle. (Each commander can defend or attack only one castle at a time.)
(2) For any arbitrary set X of castles in country A, the number of country A's castles that are in X or adjacent to at least one of the castle in X is not less than the number of country B's castles that are adjacent to at least one of the castles in X.