Croatia Team Selection Test 2007

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Day 1

1 Find integral solutions to the equation

$$(m^2 - n^2)^2 = 16n + 1.$$

2 Prove that the sequence $a_n = \lfloor n\sqrt{2} \rfloor + \lfloor n\sqrt{3} \rfloor$ contains infintely many even and infinitely many odd numbers.

3 Let ABC be a triangle such that $|AC| > |AB|$. Let X be on line AB (closer to A) such that $|BX| = |AC|$ and let Y be on the segment AC such that $|CY| = |AB|$. Intersection of lines XY and bisector of BC is point P. Prove that $\angle BPC + \angle BAC = 180^\circ$.

4 Given a finite string S of symbols X and O, we write $@(S)$ for the number of X's in S minus the number of O's. (For example, $@(XOOXOOX) = -1$.) We call a string S **balanced** if every substring T of (consecutive symbols) S has the property $-2 \leq @(T) \leq 2$. (Thus $XOOXOOX$ is not balanced since it contains the sub-string $OOXOO$ whose $@$-value is -3.) Find, with proof, the number of balanced strings of length n.

Day 2

5 Let there be two circles. Find all points M such that there exist two points, one on each circle such that M is their midpoint.

6 $2n$ students $(n \geq 5)$ participated at table tennis contest, which took 4 days. In every day, every student played a match. (It is possible that the same pair meets twice or more times, in different days) Prove that it is possible that the contest ends like this:

- there is only one winner;

- there are 3 students on the second place;

- no student lost all 4 matches.

How many students won only a single match and how many won exactly 2 matches? (In the above conditions)

7 Let $a, b, c > 0$ such that $a+b+c=1$. Prove:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3(a^2+b^2+c^2)$$

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8 Positive integers $x>1$ and y satisfy an equation $2x^2-1=y^{15}$. Prove that 5 divides x.
