Croatia Team Selection Test 2011

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Day 1

1 Let a,b,c be positive reals such that $a+b+c=3$. Prove the inequality

$$\frac{a^2}{a+b^2}+\frac{b^2}{b+c^2}+\frac{c^2}{c+a^2}\geq\frac{3}{2}.$$

2 There were finitely many persons at a party among whom some were friends. Among any 4 of them there were either 3 who were all friends among each other or 3 who weren't friend with each other. Prove that you can separate all the people at the party in two groups in such a way that in the first group everyone is friends with each other and that all the people in the second group are not friends to anyone else in second group. (Friendship is a mutual relation).

3 Triangle ABC is given with its centroid G and cicumcentre O is such that GO is perpendicular to AG. Let A' be the second intersection of AG with circumcircle of triangle ABC. Let D be the intersection of lines CA' and AB and E the intersection of lines BA' and AC. Prove that the circumcentre of triangle ADE is on the circumcircle of triangle ABC.

4 We define the sequence x_n so that

$$x_1=a, x_2=b, x_n=\frac{{x_{n-1}}^2+{x_{n-2}}^2}{x_{n-1}+x_{n-2}}\quad\forall n\geq 3.$$

Where $a,b>1$ are relatively prime numbers. Show that x_n is not an integer for $n\geq 3$.

Day 2

1 We define a sequence a_n so that $a_0=1$ and

$$a_{n+1}=\begin{cases}\frac{a_n}{2} & \text{if } a_n\equiv 0 \pmod 2,\\ a_n+d & \text{otherwise.}\end{cases}$$

for all postive integers n.
Find all positive integers d such that there is some positive integer i for which $a_i=1$.

2 There are lamps in every field of $n\times n$ table. At start all the lamps are off. A move consists of chosing m consecutive fields in a row or a column and changing the status of that m lamps. Prove that you can reach a state in which all the lamps are on only if m divides n.

3 Let K and L be the points on the semicircle with diameter AB. Denote intersection of AK and AL as T and let N be the point such that N is on segment AB and line TN is perpendicular to AB. If U is the intersection of perpendicular bisector of AB an KL and V is a point on KL such that angles UAV and UBV are equal. Prove that NV is perpendicular to KL.

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4 Find all pairs of integers x, y for which

$$x^3 + x^2 + x = y^2 + y.$$
