IMS 2007

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1 Suppose there exists a group with exactly n subgroups of index 2. Prove that there exists a finite abelian group G that has exactly n subgroups of index 2.

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2 Does there exist two unfair dices such that probability of their sum being j be a number in $\left(\frac{2}{33}, \frac{4}{33}\right)$ for each $2 \leq j \leq 12$?

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3 Prove that $\mathbb{R}^2$ has a dense subset such that has no three collinear points.

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4 Prove that:

$$\det(A) = \frac{1}{n!} \begin{vmatrix} \text{tr}(A) & 1 & 0 & \dots & \dots & 0 \\ \text{tr}(A^2) & \text{tr}(A) & 2 & 0 & \dots & 0 \\ \text{tr}(A^3) & \text{tr}(A^2) & \text{tr}(A) & 3 & & \vdots \\ \vdots & & & & & n-1 \\ \text{tr}(A^n) & \text{tr}(A^{n-1}) & \text{tr}(A^{n-2}) & \dots & \dots & \text{tr}(A) \end{vmatrix}$$

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5 Find all real α, β such that the following limit exists and is finite:

$$\lim_{x,y\to 0^+} \frac{x^{2\alpha}y^{2\beta}}{x^{2\alpha}+y^{3\beta}}$$

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6 Let R be a commutative ring with 1. Prove that $R[x]$ has infinitely many maximal ideals.

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7 $x_1, x_2, \dots, x_n$ are real number such that for each i, the set $\{x_1, x_2, \dots, x_n\}\backslash\{x_i\}$ could be partitioned into two sets that sum of elements of first set is equal to the sum of the elements of the other. Prove that all of x_i's are zero.
It is a number theory problem.

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8 Let

$$T = \{(tq, 1-t) \in \mathbb{R}^2 | t \in [0,1], q \in \mathbb{Q}\}$$

Prove that each continuous function $f : T \longrightarrow T$ has a fixed point.
