IMS 2008

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1 Let $A_1, A_2, \dots, A_n$ be idempotent matrices with real entries. Prove that:

$$\text{N}(A_1) + \text{N}(A_2) + \dots + \text{N}(A_n) \geq \text{rank}(I - A_1 A_2 \dots A_n)$$

$\text{N}(A)$ is $\dim(\text{ker(A)})$

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2 Let f be an entire function on $\mathbb{C}$ and ω_1, ω_2 are complex numbers such that $\frac{\omega_1}{\omega_2} \in \mathbb{C} \backslash \mathbb{Q}$. Prove that if for each $z \in \mathbb{C}$, $f(z) = f(z + \omega_1) = f(z + \omega_2)$ then f is constant.

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3 Let A, B be different points on a parabola. Prove that we can find $P_1, P_2, \dots, P_n$ between A, B on the parabola such that area of the convex polygon $AP_1P_2 \dots P_nB$ is maximum. In this case prove that the ratio of $S(AP_1P_2 \dots P_nB)$ to the sector between A and B doesn't depend on A and B, and only depends on n.

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4 A subset of $n \times n$ table is called even if it contains even elements of each row and each column. Find the minimum k such that each subset of this table with k elements contains an even subset

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5 Prove that there does not exist a ring with exactly 5 regular elements.
(a is called a regular element if $ax = 0$ or $xa = 0$ implies $x = 0$.)

A ring is not necessarily commutative, does not necessarily contain unity element, or is not necessarily finite.

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6 Let $a_0, a_1, \dots, a_{n+1}$ be natural numbers such that $a_0 = a_{n+1} = 1$, $a_i > 1$ for all $1 \leq i \leq n$, and for each $1 \leq j \leq n$, $a_i | a_{i-1} + a_{i+1}$. Prove that there exist one 2 in the sequence.

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7 In a contest there are n yes-no problems. We know that no two contestants have the same set of answers. To each question we give a random uniform grade of set $\{1, 2, 3, \dots, 2n\}$. Prove that the probability that exactly one person gets first is at least $\frac{1}{2}$.

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8 Find all natural numbers such that

$$n\sigma(n) \equiv 2 \pmod{\phi(n)}$$

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9 Let $\gamma : [0,1] \to [0,1] \times [0,1]$ be a mapping such that for each $s, t \in [0,1]$

$$|\gamma(s) - \gamma(t)| \leq M|s - t|^\alpha$$

in which α, M are fixed numbers. Prove that if γ is surjective, then $\alpha \leq \frac{1}{2}$