IMS 2009

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1 G is a group. Prove that the following are equivalent:
1. All subgroups of G are normal.
2. For all $a, b \in G$ there is an integer m such that $(ab)^m = ba$.

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2 Let R be a ring with 1. Every element in R can be written as product of idempotent ($u^n = u$ for some n) elements. Prove that R is commutative

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3 Let $A \subset \mathbb{C}$ be a closed and countable set. Prove that if the analytic function $f : \mathbb{C} \backslash A \longrightarrow \mathbb{C}$ is bounded, then f is constant.

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4 In this infinite tree, degree of each vertex is equal to 3. A real number λ is given. We want to assign a real number to each node in such a way that for each node sum of numbers assigned to its neighbors is equal to λ times of the number assigned to this node.
Find all λ for which this is possible.

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5 Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative and continuous function that $\iint_{\mathbb{R}^2} f(x, y)dxdy = 1$. Prove that there is a closed disc D with the least radius possible such that $\iint_D f(x, y)dxdy = \frac{1}{2}$.

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6 Suppose that there are 100 seats in a saloon for 100 students. All students except one know their seat. First student (which is the one who doesn't know his seat) comes to the saloon and sits randomly somewhere. Then others enter the saloon one by one. Every student that enters the saloon and finds his seat vacant, sits there and if he finds his seat occupied he sits somewhere else randomly. Find the probability that last two students sit on their seats.

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7 Let G be a group such that G' is abelian and each normal and abelian subgroup of G is finite. Prove that G is finite.
