IMS 2014

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by wiseman

Day 1

1 Let A be a subset of the irrational numbers such that the sum of any two distinct elements of it be a rational number. Prove that A has two elements at most.

2 Let (X,d) be a nonempty connected metric space such that the limit of every convergent sequence, is a term of that sequence. Prove that X has exactly one element.

3 Let R be a commutative ring with 1 such that the number of elements of R is equal to p^3 where p is a prime number. Prove that if the number of elements of zd(R) be in the form of p^n $(n \in \mathbb{N}^*)$ where zd$(R) = \{a \in R \mid \exists 0 \neq b \in R, ab = 0\}$, then R has exactly one maximal ideal.

4 Let (X,d) be a metric space and $f : X \to X$ be a function such that $\forall x, y \in X : d(f(x), f(y)) = d(x,y)$. a) Prove that for all $x \in X$, $\lim_{n\to+\infty} \frac{d(x,f^n(x))}{n}$ exists, where $f^n(x)$ is $\underbrace{f(f(\cdots f(x)}_{n\text{times}}\cdots))$.

b) Prove that the amount of the limit does **not** depend on choosing x.

5 Let G_1 and G_2 be two finite groups such that for any finite group H, the number of group homomorphisms from G_1 to H is equal to the number of group homomorphisms from G_2 to H. Prove that G_1 and G_2 are Isomorphic.

6 Let $A = [a_{ij}]_{n\times n}$ be a $n \times n$ matrix whose elements are all numbers which belong to set $\{1, 2, \cdots, n\}$. Prove that by swapping the columns of A with each other we can produce matrix $B = [b_{ij}]_{n\times n}$ such that $K(B) \leq n$ where $K(B)$ is the number of elements of set $\{(i,j); b_{ij} = j\}$.

Day 2

7 Let G be a finite group such that for every two subgroups of it like H and K, $H \cong K$ or $H \subseteq K$ or $K \subseteq H$. Prove that we can produce each subgroup of G with 2 elements at most.

8 Is $\sum_{n=1}^{+\infty} \frac{\cos n}{n}(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}})$ convergent? why?

9 Let G be a $2n-$vertices simple graph such that in any partition of the set of vertices of G into two $n-$vertices sets V_1 and V_2, the number of edges from a vertex in V_1 to another vertex in V_1 is equal to the number of edges from a vertex in V_2 to another vertex in V_2. Prove that all the vertices have equal degrees.

10 Let V be a $n-$dimensional vector space over a field F with a basis $\{e_1, e_2, \cdots, e_n\}$.Prove that for any $m-$dimensional linear subspace W of V, the number of elements of the set $W \cap P$ is less than or equal to 2^m where $P = \{\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n : \lambda_i = 0, 1\}$.

11 Let the equation $a^2 + b^2 + 1 = abc$ have answer in $\mathbb{N}$.Prove that $c = 3$.

12 Let U be an open subset of the complex plane $\mathbb{C}$ including $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and f be analytic over U. Prove that if for every z with a complex norm equal to $1(|z| = 1)$ we have $0 < Re(\bar{z}f(z))$, then f has only one root in $\mathbb{D}$ and that's simple.