Macedonia National Olympiad 2006

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by WakeUp

1 A natural number is written on the blackboard. In each step, we erase the units digit and add four times the erased digit to the remaining number, and write the result on the blackboard instead of the initial number. Starting with the number 13^{2006}, is it possible to obtain the number 2006^{13} by repeating this step finitely many times?

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2 Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all x,y,z,

$$f(x+y^2+z)=f(f(x))+yf(y)+f(z).$$

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3 Let a,b,c be real numbers distinct from 0 and 1, with $a+b+c=1$. Prove that

$$8\left(\frac{1}{2}-ab-bc-ca\right)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\geq 9$$

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4 Let M be a point on the smaller arc A_1A_n of the circumcircle of a regular n-gon $A_1A_2\ldots A_n$.

(a) If n is even, prove that $\sum_{i=1}^{n}(-1)^iMA_i^2=0$.

(b) If n is odd, prove that $\sum_{i=1}^{n}(-1)^iMA_i=0$.

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5 All segments joining n points (no three of which are collinear) are coloured in one of k colours. What is the smallest k for which there always exists a closed polygonal line with the vertices at some of the n points, whose sides are all of the same colour?
