Macedonia National Olympiad 2007

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1 Let a, b, c be positive real numbers. Prove that

$$1+\frac{3}{ab+bc+ca}\geq\frac{6}{a+b+c}.$$

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2 In a trapezoid $ABCD$ with a base AD, point L is the orthogonal projection of C on AB, and K is the point on BC such that AK is perpendicular to AD. Let O be the circumcenter of triangle ACD. Suppose that the lines AK, CL and DO have a common point. Prove that $ABCD$ is a parallelogram.

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3 Natural numbers a, b and c are pairwise distinct and satisfy

$$a|b+c+bc, b|c+a+ca, c|a+b+ab.$$

Prove that at least one of the numbers a, b, c is not prime.

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4 Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(x^3+y^3)=x^2f(x)+yf(y^2)$$

for all $x, y\in\mathbb{R}$.

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5 Let n be a natural number divisible by 4. Determine the number of bijections f on the set $\{1,2,...,n\}$ such that $f(j)+f^{-1}(j)=n+1$ for $j=1,...,n$.
