Macedonia National Olympiad 2008

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by April, test20

1 Find all injective functions $f:\mathbb{N}\to\mathbb{N}$ which satisfy

$$f(f(n))\leq\frac{n+f(n)}{2}$$

for each $n\in\mathbb{N}$.

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2 Positive numbers a, b, c are such that $(a+b)\,(b+c)\,(c+a)=8$. Prove the inequality

$$\frac{a+b+c}{3}\geq\sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

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3 An acute triangle ABC with $AB\neq AC$ is given. Let V and D be the feet of the altitude and angle bisector from A, and let E and F be the intersection points of the circumcircle of $\triangle AVD$ with sides AC and AB, respectively. Prove that AD, BE and CF have a common point.

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4 We call an integer $n>1$ *good* if, for any natural numbers $1\leq b_1,b_2,\ldots,b_{n-1}\leq n-1$ and any $i\in\{0,1,\ldots,n-1\}$, there is a subset I of $\{1,\ldots,n-1\}$ such that $\sum_{k\in I}b_k\equiv i\pmod n$. (The sum over the empty set is zero.) Find all good numbers.
