Macedonia National Olympiad 2009

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1 Find all natural numbers x, y, z such that $a + 2^x 3^y = z^2$.

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2 Let O be the centre of the incircle of $\triangle ABC$. Points K, L are the intersection points of the circles circumscribed about triangles BOC, AOC respectively with the bisectors of the angles at A, B respectively $(K, L \neq O)$. Also P is the midpoint of segment KL, M is the reflection of O with respect to P and N is the reflection of O with respect to line KL. Prove that the points K, L, M and N lie on the same circle.

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3 The Macedonian Mathematical Olympiad is held in two rooms numbered 1 and 2. At the beginning all of the competitors enter room No. 1. The final arrangement of the competitors to the rooms is obtained in the following way: a list with the names of a few of the competitors is read aloud; after a name is read, the corresponding competitor and all of his/her acquaintances from the rest of the competitors change the room in which they currently are. Hence, to each list of names corresponds one final arrangement of the competitors to the rooms. Show that the total number of possible final arrangements is not equal to 2009 (acquaintance between competitors is a symmetrical relation).

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4 Let a, b, c be positive real numbers for which $ab + bc + ca = \frac{1}{3}$. Prove the inequality

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}$$

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5 Solve the following equation in the set of integer numbers:

$$x^{2010} - 2006 = 4y^{2009} + 4y^{2008} + 2007y.$$
