Macedonia National Olympiad 2011

www.artofproblemsolving.com/community/c3573

by StefanS

1 Let $a, b, c, d > 0$ and $a+b+c+d = 1$. Prove the inequality

$$\frac{1}{4a+3b+c}+\frac{1}{3a+b+4d}+\frac{1}{a+4c+3d}+\frac{1}{4b+3c+d} \geq 2\,.$$

---

2 Acute-angled $\triangle ABC$ is given. A line l parallel to side AB passing through vertex C is drawn. Let the angle bisectors of $\angle BAC$ and $\angle ABC$ intersect the sides BC and AC at points D and F, and line l at points E and G respectively. Prove that if $\overline{DE} = \overline{GF}$ then $\overline{AC} = \overline{BC}$.

---

3 Find all natural numbers n for which each natural number written with $n-1$ 'ones' and one 'seven' is prime.

---

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+yf(x)) = f(f(x)) + xf(y)\,.$$

---

5 A table of the type $(n_1, n_2, ..., n_m)$, $n_1 \geq n_2 \geq ... \geq n_m$ is defined in the following way: n_1 squares are ordered horizontally one next to another, then n_2 squares are ordered horizontally beneath the already ordered n_1 squares. The procedure continues until a net composed of n_1 squares in the first row, n_2 in the second, n_i in the i-th row is obtained, such that there are totally $n = n_1 + n_2 + ... + n_m$ squares in the net. The ordered rows form a straight line on the left, as shown in the example. The obtained table is filled with the numbers from 1 till n in a way that the numbers in each row and column become greater from left to right and from top to bottom, respectively. An example of a table of the type $(5,4,2,1)$ and one possible way of filling it is attached to the post. Find the number of ways the table of type $(4,3,2)$ can be filled.
