Macedonia National Olympiad 2012

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by StefanS

1 Solve the equation $x^4+2y^4+4z^4+8t^4=16xyzt$ in the set of integer numbers.

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2 If $a,\,b,\,c,\,d$ are positive real numbers such that $abcd=1$ then prove that the following inequality holds

$$\frac{1}{bc+cd+da-1}+\frac{1}{ab+cd+da-1}+\frac{1}{ab+bc+da-1}+\frac{1}{ab+bc+cd-1}\;\leq\;2\,.$$

When does inequality hold?

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3 Find all functions $f:\mathbb{R}\to\mathbb{Z}$ which satisfy the conditions:

$f(x+y)<f(x)+f(y)$

$f(f(x))=\lfloor x\rfloor+2$

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4 A fixed circle k and collinear points E,F and G are given such that the points E and G lie outside the circle k and F lies inside the circle k. Prove that, if $ABCD$ is an arbitrary quadrilateral inscribed in the circle k such that the points E,F and G lie on lines AB,AD and DC respectively, then the side BC passes through a fixed point collinear with E,F and G, independent of the quadrilateral $ABCD$.

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5 A hexagonal table is given, as the one on the drawing, which has 2012 columns. There are 2012 hexagons in each of the odd columns, and there are 2013 hexagons in each of the even columns. The number i is written in each hexagon from the i-th column. Changing the numbers in the table is allowed in the following way: We arbitrarily select three adjacent hexagons, we rotate the numbers, and if the rotation is clockwise then the three numbers decrease by one, and if we rotate them counterclockwise the three numbers increase by one (see the drawing below). What's the maximum number of zeros that can be obtained in the table by using the above-defined steps.
