Northern Mathematical Olympiad 2007

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by April

Day 1

1 Let ABC be acute triangle. The circle with diameter AB intersects CA, CB at M, N, respectively. Draw $CT \perp AB$ and intersects above circle at T, where C and T lie on the same side of AB. S is a point on AN such that $BT = BS$. Prove that $BS \perp SC$.

2 Let $a,\ b,\ c$ be side lengths of a triangle and $a+b+c=3$. Find the minimum of

$$a^2+b^2+c^2+\frac{4abc}{3}$$

3 Sequence $\{a_n\}$ is defined by $a_1=2007,\ a_{n+1}=\frac{a_n^2}{a_n+1}$ for $n\geq 1$. Prove that $[a_n]=2007-n$ for $0\leq n\leq 1004$, where $[x]$ denotes the largest integer no larger than x.

4 For every point on the plane, one of n colors are colored to it such that:
(1) Every color is used infinitely many times.
(2) There exists one line such that all points on this lines are colored exactly by one of two colors.

Find the least value of n such that there exist four concyclic points with pairwise distinct colors.

Day 2

1 Let α, β be acute angles. Find the maximum value of

$$\frac{\left(1-\sqrt{\tan\alpha\tan\beta}\right)^2}{\cot\alpha+\cot\beta}$$

2 Let f be a function given by $f(x)=\lg(x+1)-\frac{1}{2}\cdot\log_3 x$.

a) Solve the equation $f(x)=0$.

b) Find the number of the subsets of the set

$$\{n|f(n^2-214n-1998)\geq 0,\ n\in\mathbb{Z}\}.$$

3 Let n be a positive integer and $[\,n]=a$. Find the largest integer n such that the following two conditions are satisfied:
(1) n is not a perfect square;
(2) a^3 divides n^2.

4 The inradius of triangle ABC is 1 and the side lengths of ABC are all integers. Prove that triangle ABC is right-angled.