Northern Mathematical Olympiad 2013

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1 Find the largest positive integer n ($n \geq 3$), so that there is a convex n-gon, the tangent of each interior angle is an integer.

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2 If $a_1, a_2, \cdots, a_{2013} \in [-2, 2]$ and $a_1 + a_2 + \cdots + a_{2013} = 0$, find the maximum of $a_1^3 + a_2^3 + \cdots + a_{2013}^3$.

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3 As shown in figure , A, B are two fixed points of circle $\odot O$, C is the midpoint of the major arc AB, D is any point of the minor arc AB. Tangent at D intersects tangents at A, B at points E, F respectively. Segments CE and CF intersect chord AB at points G and H respectively. Prove that the length of line segment GH has a fixed value.
`https://cdn.artofproblemsolving.com/attachments/9/2/85227f169193f61e313293e9128f6ece2ff1f`
`png`

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4 For positive integers n, a, b, if $n = a^2 + b^2$, and a and b are coprime, then the number pair (a, b) is called a *square split* of n (the order of a, b does not count). Prove that for any positive k, there are only two square splits of the integer 13^k.

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5 Find all non-integers x such that $x + \frac{13}{x} = [x] + \frac{13}{[x]}$.where$[x]$ mean the greatest integer n , where $n \leq x.$

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6 As shown in figure , it is known that M is the midpoint of side BC of $\triangle ABC$. $\odot O$ passes through points A, C and is tangent to AM. The extension of the segment BA intersects $\odot O$ at point D. The lines CD and MA intersect at the point P. Prove that $PO \perp BC$.
`https://cdn.artofproblemsolving.com/attachments/8/a/da3570ec7eb0833c7a396e22ffac2bd890218`
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7 Suppose that $\{a_n\}$ is a sequence such that $a_{n+1} = (1 + \frac{k}{n})a_n + 1$ with $a_1 = 1$.Find all positive integers k such that any a_n be integer.

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8 $3n$ $(n \geq 2, n \in N)$ people attend a gathering, in which any two acquaintances have exactly n common acquaintances, and any two unknown people have exactly $2n$ common acquaintances. If three people know each other, it is called a *Taoyuan Group*.
(1) Find the number of all Taoyuan groups;
(2) Prove that these $3n$ people can be divided into three groups, with n people in each group, and the three people obtained by randomly selecting one person from each group constitute a Taoyuan group.

Note: Acquaintance means that two people know each other, otherwise they are not acquaintances. Two people who know each other are called acquaintances.