Northern Mathematical Olympiad 2014

www.artofproblemsolving.com/community/c3579

by parmenides51, sqing

1   As shown in the figure, given $\triangle ABC$ with $\angle B$, $\angle C$ acute angles, $AD \perp BC$, $DE \perp AC$, M midpoint of DE, $AM \perp BE$. Prove that $\triangle ABC$ is isosceles.

https://cdn.artofproblemsolving.com/attachments/a/8/f553c33557979f6f7b799935c3bde743edcc3 png

2   Define a positive number sequence sequence $\{a_n\}$ by

$$a_1 = 1, (n^2 + 1)a_{n-1}^2 = (n-1)^2 a_n^2.$$

Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \leq 1 + \sqrt{1 - \frac{1}{a_n^2}}.$$

3   Determine whether there exist an infinite number of positive integers x, y satisfying the condition: $x^2 + y \mid x + y^2$. Please prove it.

4   In an election, there are a total of 12 candidates. An election committee has 6 members voting. It is known that at most two candidates voted by any two committee members are the same. Find the maximum number of committee members.

5   As shown in the figure, in the parallelogram $ABCD$, I is the incenter of $\triangle BCD$, and H is the orthocenter of $\triangle IBD$. Prove that $\angle HAB = \angle HAD$.

https://cdn.artofproblemsolving.com/attachments/4/3/5fa16c208ef3940443854756ae7bdb9c4272e png

6   Let x, y, z, w be real numbers such that $x + 2y + 3z + 4w = 1$. Find the minimum of $x^2 + y^2 + z^2 + w^2 + (x + y + z + w)^2$.

7   Prove that there exist infinitely many positive integers n such that $3^n + 2$ and $5^n + 2$ are all composite numbers.

8   Two people, A and B, play the game of blowing up a balloon. The balloon will explode only when the volume of the balloon $V > 2014$ mL. A blows in 1 mL first, and then they takes turns blowing. It is agreed that the gas blown by each person must not be less than the gas blown by the other party last time and should not be more than twice the amount of gas the other party blew last time. The agreement is that the person who blows up the balloon loses. Who has a winning

strategy ? Briefly explain it. (Do not consider the change in volume caused by the change in tension when the balloon is inflated).