Middle European Mathematical Olympiad 2007

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Day 1

1 Let a, b, c, d be positive real numbers with $a+b+c+d=4$.
Prove that
$$a^2bc+b^2cd+c^2da+d^2ab \leq 4.$$

2 A set of balls contains n balls which are labeled with numbers $1, 2, 3, \ldots, n$. We are given $k > 1$ such sets. We want to colour the balls with two colours, black and white in such a way, that

(a) the balls labeled with the same number are of the same colour,

(b) any subset of $k+1$ balls with (not necessarily different) labels $a_1, a_2, \ldots, a_{k+1}$ satisfying the condition $a_1+a_2+\ldots+a_k = a_{k+1}$, contains at least one ball of each colour.

Find, depending on k the greatest possible number n which admits such a colouring.

3 Let k be a circle and k_1, k_2, k_3, k_4 four smaller circles with their centres O_1, O_2, O_3, O_4 respectively, on k. For $i = 1, 2, 3, 4$ and $k_5 = k_1$ the circles k_i and k_{i+1} meet at A_i and B_i such that A_i lies on k. The points $O_1, A_1, O_2, A_2, O_3, A_3, O_4, A_4$ lie in that order on k and are pairwise different.

Prove that $B_1B_2B_3B_4$ is a rectangle.

4 Determine all pairs (x, y) of positive integers satisfying the equation
$$x!+y! = x^y.$$

Day 2

1 Let a, b, c, d be real numbers which satisfy $\frac{1}{2} \leq a, b, c, d \leq 2$ and $abcd = 1$. Find the maximum value of
$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right).$$

2 For a set P of five points in the plane, no three of them being collinear, let $s(P)$ be the numbers of acute triangles formed by vertices in P.
Find the maximum value of $s(P)$ over all such sets P.

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3 A tetrahedron is called a *MEMO-tetrahedron* if all six sidelengths are different positive integers where one of them is 2 and one of them is 3. Let $l(T)$ be the sum of the sidelengths of the tetrahedron T.
(a) Find all positive integers n so that there exists a MEMO-Tetrahedron T with $l(T) = n$.
(b) How many pairwise non-congruent MEMO-tetrahedrons T satisfying $l(T) = 2007$ exist? Two tetrahedrons are said to be non-congruent if one cannot be obtained from the other by a composition of reflections in planes, translations and rotations. (It is not neccessary to prove that the tetrahedrons are not degenerate, i.e. that they have a positive volume).

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4 Find all positive integers k with the following property: There exists an integer a so that $(a+k)^3 - a^3$ is a multiple of 2007.
