Middle European Mathematical Olympiad 2008

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by orl

Day 1

1 Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers with $a_n < a_{n+1}, \quad \forall n \geq 1$. For all quadruple (i,j,k,l) of indices such that $1 \leq i < j \leq k < l$ and $i+l = j+k$ we have the inequality $a_i + a_l > a_j + a_k$. Determine the least possible value of a_{2008}.

2 Consider a $n \times n$ checkerboard with $n > 1, n \in \mathbb{N}$. How many possibilities are there to put $2n-2$ identical pebbles on the checkerboard (each on a different field/place) such that no two pebbles are on the same checkerboard diagonal. Two pebbles are on the same checkerboard diagonal if the connection segment of the midpoints of the respective fields are parallel to one of the diagonals of the $n \times n$ square.

3 Let ABC be an isosceles triangle with $AC = BC$. Its incircle touches AB in D and BC in E. A line distinct of AE goes through A and intersects the incircle in F and G. Line AB intersects line EF and EG in K and L, respectively. Prove that $DK = DL$.

4 Determine that all $k \in \mathbb{Z}$ such that $\forall n$ the numbers $4n+1$ and $kn+1$ have no common divisor.

Day 2

1 Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$xf(x+xy) = xf(x) + f\left(x^2\right)f(y) \quad \forall x, y \in \mathbb{R}.$$

2 On a blackboard there are $n \geq 2, n \in \mathbb{Z}^+$ numbers. In each step we select two numbers from the blackboard and replace both of them by their sum. Determine all numbers n for which it is possible to yield n identical number after a finite number of steps.

3 Let ABC be an acute-angled triangle. Let E be a point such E and B are on distinct sides of the line AC, and D is an interior point of segment AE. We have $\angle ADB = \angle CDE$, $\angle BAD = \angle ECD$, and $\angle ACB = \angle EBA$. Prove that B, C and E lie on the same line.

4 Prove: If the sum of all positive divisors of $n \in \mathbb{Z}^+$ is a power of two, then the number/amount of the divisors is a power of two.