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– **Red Group**

– **Algebra**

1 Let  $a$  and  $b$  be nonnegative real numbers. Prove that

$$\sqrt{2} \left( \sqrt{a(a+b)^3} + b\sqrt{a^2+b^2} \right) \leq 3(a^2+b^2).$$

2 Determine if there exist four polynomials such that the sum of any three of them has a real root while the sum of any two of them does not.

3 Let  $a, b, c$  be real numbers. Prove that

$$\begin{aligned} & \sqrt{2(a^2+b^2)} + \sqrt{2(b^2+c^2)} + \sqrt{2(c^2+a^2)} \\ & \geq \sqrt{3[(a+b)^2 + (b+c)^2 + (c+a)^2]}. \end{aligned}$$

4 Let  $x_1, x_2, \dots, x_5$  be nonnegative real numbers such that  $x_1 + x_2 + x_3 + x_4 + x_5 = 5$ . Determine the maximum value of  $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5$ .

5 Let  $S$  be a finite set of positive integers such that none of them has a prime factor greater than three. Show that the sum of the reciprocals of the elements in  $S$  is smaller than three.

6 Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{R}$  such that  $f(1) = \frac{5}{2}$  and that

$$f(x)f(y) = f(x+y) + f(x-y)$$

for all integers  $x$  and  $y$ .

7 Let  $x_{1,1}, x_{2,1}, \dots, x_{n,1}$ ,  $n \geq 2$ , be a sequence of integers and assume that not all  $x_{i,1}$  are equal. For  $k \geq 2$ , if sequence  $\{x_{i,k}\}_{i=1}^n$  is defined, we define sequence  $\{x_{i,k+1}\}_{i=1}^n$  as

$$x_{i,k+1} = \frac{1}{2}(x_{i,k} + x_{i+1,k}),$$

for  $i = 1, 2, \dots, n$ , (where  $x_{n+1,k} = x_{1,k}$ ). Show that if  $n$  is odd then there exist indices  $j$  and  $k$  such that  $x_{j,k}$  is not an integer.

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**– Combinatorics**

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**1** Two rooks on a chessboard are said to be attacking each other if they are placed in the same row or column of the board.

(a) There are eight rooks on a chessboard, none of them attacks any other. Prove that there is an even number of rooks on black fields.

(b) How many ways can eight mutually non-attacking rooks be placed on the 9 9 chessboard so that all eight rooks are on squares of the same color.

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**2** Set  $S = \{1, 2, \dots, 2004\}$ . We denote by  $d_1$  the number of subset of  $S$  such that the sum of elements of the subset has remainder 7 when divided by 32. We denote by  $d_2$  the number of subset of  $S$  such that the sum of elements of the subset has remainder 14 when divided by 16. Compute  $\frac{d_1}{d_2}$ .

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**3** In a television series about incidents in a conspicuous town there are  $n$  citizens staging in it, where  $n$  is an integer greater than 3. Each two citizens plan together a conspiracy against one of the other citizens. Prove that there exists a citizen, against whom at least  $\sqrt{n}$  other citizens are involved in the conspiracy.

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**4** Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there are three players A,B, and C such that A beats B, B beats C, and C beats A. Such a triple of player is called a cycle. Determine the number of maximum cycles such a tournament can have.

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**5** Determine if it is possible to choose nine points in the plane such that there are  $n = 10$  lines in the plane each of which passes through exactly three of the chosen points. What if  $n = 11$ ?

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**6** Let  $n$  be a positive integer. Show that

$$\begin{aligned} & \frac{1}{\binom{n}{1}} + \frac{1}{2\binom{n}{2}} + \frac{1}{3\binom{n}{3}} + \cdots + \frac{1}{n\binom{n}{n}} \\ &= \frac{1}{2^{n-1}} + \frac{1}{2 \cdot 2^{n-2}} + \frac{1}{3 \cdot 2^{n-3}} + \cdots + \frac{1}{n \cdot 2^0}. \end{aligned}$$

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**7** A segment of length 2 is divided into  $n$ ,  $n \geq 2$ , subintervals. A square is then constructed on each subinterval. Assume that the sum of the areas of all such squares is greater than 1. Show that under this assumption one can always choose two subintervals with total length greater than 1.

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**– Geometry**

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1 Isosceles triangle  $ABC$ , with  $AB = AC$ , is inscribed in circle  $\omega$ . Point  $D$  lies on arc  $\widehat{BC}$  not containing  $A$ . Let  $E$  be the foot of perpendicular from  $A$  to line  $CD$ . Prove that  $BC + DC = 2DE$ .

2 Let  $ABC$  be a triangle, and let  $D$  be a point on side  $AB$ . Circle  $\omega_1$  passes through  $A$  and  $D$  and is tangent to line  $AC$  at  $A$ . Circle  $\omega_2$  passes through  $B$  and  $D$  and is tangent to line  $BC$  at  $B$ . Circles  $\omega_1$  and  $\omega_2$  meet at  $D$  and  $E$ . Point  $F$  is the reflection of  $C$  across the perpendicular bisector of  $AB$ . Prove that points  $D$ ,  $E$ , and  $F$  are collinear.

3 Let  $M$  be the midpoint of side  $BC$  of triangle  $ABC$  ( $AB > AC$ ), and let  $AL$  be the bisector of the angle  $A$ . The line passing through  $M$  perpendicular to  $AL$  intersects the side  $AB$  at the point  $D$ . Prove that  $AD + MC$  is equal to half the perimeter of triangle  $ABC$ .

4 Let  $ABC$  be an obtuse triangle with  $\angle A > 90^\circ$ , and let  $r$  and  $R$  denote its inradius and circumradius. Prove that

$$\frac{r}{R} \leq \frac{a \sin A}{a + b + c}.$$

5 Let  $ABC$  be a triangle. Points  $D$  and  $E$  lie on sides  $BC$  and  $CA$ , respectively, such that  $BD = AE$ . Segments  $AD$  and  $BE$  meet at  $P$ . The bisector of angle  $BCA$  meet segments  $AD$  and  $BE$  at  $Q$  and  $R$ , respectively. Prove that  $\frac{PQ}{AD} = \frac{PR}{BE}$ .

6 Consider the three disjoint arcs of a circle determined by three points of the circle. We construct a circle around each of the midpoint of every arc which goes the end points of the arc. Prove that the three circles pass through a common point.

7 Let  $ABC$  be a triangle. Prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 4 \left( \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right).$$

## – Number Theory

1 Find all triples  $(x, y, z)$  such that  $x^2 + y^2 + z^2 = 2^{2004}$ .

2 Suppose that  $n$  is a positive integer. Determine all the possible values of the first digit after the decimal point in the decimal expression of the number  $\sqrt{n^3 + 2n^2 + n}$ .

3 Suppose that  $p$  and  $q$  are distinct primes and  $S$  is a subset of  $\{1, 2, \dots, p-1\}$ . Let  $N(S)$  denote the number of ordered  $q$ -tuples  $(x_1, x_2, \dots, x_q)$  with  $x_i \in S$ ,  $1 \leq i \leq q$ , such that  $x_1 + x_2 + \dots + x_q \cong 0 \pmod{p}$ .

- 4 Let  $p$  be an odd prime. Prove that

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.$$

- 5 Find all ordered triples  $(a, b, c)$  of positive integers such that the value of the expression

$$\left(b - \frac{1}{a}\right) \left(c - \frac{1}{b}\right) \left(a - \frac{1}{c}\right)$$

is an integer.

- 6 Let  $a_1 = 0$ ,  $a_2 = 1$ , and  $a_{n+2} = a_{n+1} + a_n$  for all positive integers  $n$ . Show that there exists an increasing infinite arithmetic progression of integers, which has no number in common in the sequence  $\{a_n\}_{n \geq 0}$ .

- 7 Let  $a$ ,  $b$ , and  $c$  be pairwise distinct positive integers, which are side lengths of a triangle. There is a line which cuts both the area and the perimeter of the triangle into two equal parts. This line cuts the longest side of the triangle into two parts with ratio  $2 : 1$ . Determine  $a$ ,  $b$ , and  $c$  for which the product  $abc$  is minimal.

– **Blue Group**

– **Algebra**

- 1 Let  $a_0, a_1, \dots, a_n$  be integers, not all zero, and all at least  $-1$ . Given that  $a_0 + 2a_1 + 2^2a_2 + \dots + 2^n a_n = 0$ , prove that  $a_0 + a_1 + \dots + a_n > 0$ .

- 2 The sequence of real numbers  $\{a_n\}$ ,  $n \in \mathbb{N}$  satisfies the following condition:  $a_{n+1} = a_n(a_n + 2)$  for any  $n \in \mathbb{N}$ . Find all possible values for  $a_{2004}$ .

- 3 Determine all polynomials  $P(x)$  with real coefficients such that  $(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x)$ .

- 4 Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))$ .

- 5 Let  $a_1, a_2, \dots, a_{2004}$  be non-negative real numbers such that  $a_1 + \dots + a_{2004} \leq 25$ . Prove that among them there exist at least two numbers  $a_i$  and  $a_j$  ( $i \neq j$ ) such that  $|\sqrt{a_i} - \sqrt{a_j}| \leq \frac{5}{2003}$ .

- 6 Let  $c$  be a fixed positive integer, and  $\{x_k\}_{k=1}^{\infty}$  be a sequence such that  $x_1 = c$  and  $x_n = x_{n-1} + \lfloor \frac{2x_{n-1} - 2}{n} \rfloor$

for  $n \geq 2$ . Determine the explicit formula of  $x_n$  in terms of  $n$  and  $c$ .  
(Here  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$ .)

- 7 Let  $n$  be a positive integer with  $n > 1$ , and let  $a_1, a_2, \dots, a_n$  be positive integers such that  $a_1 < a_2 < \dots < a_n$  and  $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \leq 1$ .

Prove that  $(\frac{1}{a_1^2+x^2} + \frac{1}{a_2^2+x^2} + \dots + \frac{1}{a_n^2+x^2})^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1-1)+x^2}$

for all real numbers  $x$ .

– **Combinatorics**

- 1 Consider all binary sequences (sequences consisting of 0s and 1s). In such a sequence the following four types of operation are allowed: (a)  $010 \rightarrow 1$ , (b)  $1 \rightarrow 010$ , (c)  $110 \rightarrow 0$ , and (d)  $0 \rightarrow 110$ . Determine if it is possible to obtain the sequence  $100\dots 0$  (with 2003 zeroes) from the sequence  $0\dots 01$  (with 2003 zeroes).

- 2 Exactly one integer is written in each square of an  $n$  by  $n$  grid,  $n \geq 3$ . The sum of all of the numbers in any  $2 \times 2$  square is even and the sum of all the numbers in any  $3 \times 3$  square is even. Find all  $n$  for which the sum of all the numbers in the grid is necessarily even.

- 3 Let  $T$  be the set of all positive integer divisors of  $2004_{100}$ . What is the largest possible number of elements that a subset  $S$  of  $T$  can have if no element of  $S$  is an integer multiple of any other element of  $S$ ?

- 4 Consider an infinite array of integers. Assume that each integer is equal to the sum of the integers immediately above and immediately to the left. Assume that there exists a row  $R_0$  such that all the number in the row are positive. Denote by  $R_1$  the row below row  $R_0$ , by  $R_2$  the row below row  $R_1$ , and so on. Show that for each positive integer  $n$ , row  $R_n$  cannot contain more than  $n$  zeros.

- 5 Show that for nonnegative integers  $m$  and  $n$ ,

$$\frac{\binom{m}{0}}{n+1} - \frac{\binom{m}{1}}{n+2} + \dots + (-1)^m \frac{\binom{m}{m}}{n+m+1} = \frac{\binom{n}{0}}{m+1} - \frac{\binom{n}{1}}{m+2} + \dots + (-1)^n \frac{\binom{n}{n}}{m+n+1}.$$

- 6 A  $10 \times 10 \times 10$  cube is made up up from 500 white unit cubes and 500 black unit cubes, arranged in such a way that every two unit cubes that shares a face are in different colors. A line is a  $1 \times 1 \times 10$  portion of the cube that is parallel to one of cubes edges. From the initial cube have been removed 100 unit cubes such that 300 lines of the cube has exactly one missing cube.

Determine if it is possible that the number of removed black unit cubes is divisible by 4.

- 7 Let  $S$  be a set of points in the plane satisfying the following conditions:  
 (a) there are seven points in  $S$  that form a convex heptagon; and  
 (b) for any five points in  $S$ , if they form a convex pentagon, then there is point in  $S$  lies in the interior of the pentagon. Determine the minimum value of the number of elements in  $S$ .

– **Geometry**

- 1 In convex hexagon  $ABCDEF$  all sides have equal length and  
 $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$ .  
 Prove that the diagonals  $AD, BE, CF$  are concurrent.
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- 2 In a convex quadrilateral  $ABCD$  the points  $P$  and  $Q$  are chosen on the sides  $BC$  and  $CD$  respectively so that  $\angle BAP = \angle DAQ$ . Prove that the line, passing through the orthocenters of triangles  $ABP$  and  $ADQ$ , is perpendicular to  $AC$  if and only if the triangles  $ABP$  and  $ADQ$  have the same areas.
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- 3 Circles  $S_1$  and  $S_2$  meet at points  $A$  and  $B$ . A line through  $A$  is parallel to the line through the centers of  $S_1$  and  $S_2$  and meets  $S_1$  and  $S_2$  again  $C$  and  $D$  respectively. Circle  $S_3$  having  $CD$  as its diameter meets  $S_1$  and  $S_2$  again at  $P$  and  $Q$  respectively. Prove that lines  $CP, DQ$ , and  $AB$  are concurrent.
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- 4 The incenter  $O$  of an isosceles triangle  $ABC$  with  $AB = AC$  meets  $BC, CA, AB$  at  $K, L, M$  respectively. Let  $N$  be the intersection of lines  $OL$  and  $KM$  and let  $Q$  be the intersection of lines  $BN$  and  $CA$ . Let  $P$  be the foot of the perpendicular from  $A$  to  $BQ$ . If we assume that  $BP = AP + 2PQ$ , what are the possible values of  $\frac{AB}{BC}$ ?
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- 5 Let  $ABCD$  be a cyclic quadrilateral such that  $AB \cdot BC = 2 \cdot AD \cdot DC$ . Prove that its diagonals  $AC$  and  $BD$  satisfy the inequality  $8BD^2 \leq 9AC^2$ .  
 Moderator says: Use the search before posting contest problems  
<http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=530783>
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- 6 A circle which is tangent to sides  $AB$  and  $BC$  of triangle  $ABC$  is also tangent to its circumcircle at point  $T$ . If  $I$  is the incenter of triangle  $ABC$ , show that  $\angle ATI = \angle CTI$ .
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- 7 Points  $E, F, G$ , and  $H$  lie on sides  $AB, BC, CD$ , and  $DA$  of a convex quadrilateral  $ABCD$  such that  $\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CG}{GD} \cdot \frac{DH}{HA} = 1$ .  
 Points  $A, B, C$ , and  $D$  lie on sides  $H_1E_1, E_1F_1, F_1G_1$ , and  $G_1H_1$  of a convex quadrilateral  $E_1F_1G_1H_1$  such that  $E_1F_1 \parallel EF, F_1G_1 \parallel FG, G_1H_1 \parallel GH$ , and  $H_1E_1 \parallel HE$ . Given that  $\frac{E_1A}{AH_1} = a$ , express  $\frac{F_1C}{CG_1}$  in terms of  $a$ .

– **Number Theory**

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- 1** We call a natural number 3-partite if the set of its divisors can be partitioned into 3 subsets each with the same sum. Show that there exist infinitely many 3-partite numbers.
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- 2** Find all real numbers  $x$  such that  $\lfloor x^2 - 2x \rfloor + 2\lfloor x \rfloor = \lfloor x \rfloor^2$ .  
(For a real number  $x$ ,  $\lfloor x \rfloor$  denote the greatest integer less than or equal to  $x$ .)
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- 3** Prove that the equation  $a^3 - b^3 = 2004$  does not have any solutions in positive integers.
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- 4** Find all prime numbers  $p$  and  $q$  such that  $3p^4 + 5q^4 + 15 = 13p^2q^2$ .
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- 5** Does there exist an infinite subset  $S$  of the natural numbers such that for every  $a, b \in S$ , the number  $(ab)^2$  is divisible by  $a^2 - ab + b^2$ ?
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- 6** A positive integer  $n$  is good if  $n$  can be written as the sum of 2004 positive integers  $a_1, a_2, \dots, a_{2004}$  such that  $1 \leq a_1 < a_2 < \dots < a_{2004}$  and  $a_i$  divides  $a_{i+1}$  for  $i = 1, 2, \dots, 2003$ . Show that there are only finitely many positive integers that are not good.
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- 7** Let  $n$  be a natural number and  $f_1, f_2, \dots, f_n$  be polynomials with integers coefficients. Show that there exists a polynomial  $g(x)$  which can be factored (with at least two terms of degree at least 1) over the integers such that  $f_i(x) + g(x)$  cannot be factored (with at least two terms of degree at least 1 over the integers for every  $i$ ).

– **Black Group**– **Algebra**

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- 1** Given real numbers  $x, y, z$  such that  $xyz = -1$ , show that  $x^4 + y^4 + z^4 + 3(x + y + z) \geq \sum_{sym} \frac{x^2}{y}$ .
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- 2** Let  $x, y, z$  be positive real numbers and  $x + y + z = 1$ . Prove that  $\sqrt{xy + z} + \sqrt{yz + x} + \sqrt{zx + y} \geq 1 + \sqrt{xy} + \sqrt{yz} + \sqrt{zx}$ .
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- 3** Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that
- (a)  $f(1) = 1$
  - (b)  $f(n + 2) + (n^2 + 4n + 3)f(n) = (2n + 5)f(n + 1)$  for all  $n \in \mathbb{N}$ .
  - (c)  $f(n)$  divides  $f(m)$  if  $m > n$ .
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- 4** Does there exist a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $x, y \in \mathbb{R}$ ,

$$f(x^2y + f(x + y^2)) = x^3 + y^3 + f(xy)$$

- 5 Find the smallest real number  $p$  such that the inequality  $\sqrt{1^2 + 1} + \sqrt{2^2 + 1} + \dots + \sqrt{n^2 + 1} \leq \frac{1}{2}n(n + p)$  holds for all natural numbers  $n$ .

- 6 Solve the system of equations:

$$x^2 = \frac{1}{y} + \frac{1}{z}, y^2 = \frac{1}{z} + \frac{1}{x}, z^2 = \frac{1}{x} + \frac{1}{y}.$$

in the real numbers.

- 7 Find all positive integers  $n$  for which there are distinct integers  $a_1, \dots, a_n$  such that

$$\frac{1}{a_1} + \frac{2}{a_2} + \dots + \frac{n}{a_n} = \frac{a_1 + a_2 + \dots + a_n}{2}.$$

– **Combinatorics**

- 1 Let  $X$  be a set with  $n$  elements and  $0 \leq k \leq n$ . Let  $a_{n,k}$  be the maximum number of permutations of the set  $X$  such that every two of them have at least  $k$  common components (where a common component of  $f$  and  $g$  is an  $x \in X$  such that  $f(x) = g(x)$ ). Let  $b_{n,k}$  be the maximum number of permutations of the set  $X$  such that every two of them have at most  $k$  common components.

(a) Show that  $a_{n,k} \cdot b_{n,k-1} \leq n!$ .

(b) Let  $p$  be prime, and find the exact value of  $a_{p,2}$ .

- 2 A regular 2004-sided polygon is given, with all of its diagonals drawn. After some sides and diagonals are removed, every vertex has at most five segments coming out of it. Prove that one can color the vertices with two colors such that at least  $\frac{3}{5}$  of the remaining segments have ends with different colors.

- 3 Squares of an  $n \times n$  table ( $n \geq 3$ ) are painted black and white as in a chessboard. A move allows one to choose any  $2 \times 2$  square and change all of its squares to the opposite color. Find all such  $n$  that there is a finite number of the moves described after which all squares are the same color.

- 4 A convex 2004-sided polygon  $P$  is given such that no four vertices are cyclic. We call a triangle whose vertices are vertices of  $P$  thick if all other 2001 vertices of  $P$  lie inside the circumcircle of the triangle, and thin if they all lie outside its circumcircle. Prove that the number of thick triangles is equal to the number of thin triangles.

- 5 A group consists of  $n$  tourists. Among every three of them there are at least two that are not familiar. For any partition of the group into two busses, there are at least two familiar tourists in one bus. Prove that there is a tourist who is familiar with at most two fifth of all the tourists.



- 6 A computer network is formed by connecting 2004 computers by cables. A set  $S$  of these computers is said to be independent if no pair of computers of  $S$  is connected by a cable. Suppose that the number of cables used is the minimum number possible such that the size of any independent set is at most 50. Let  $c(L)$  be the number of cables connected to computer  $L$ . Show that for any distinct computers  $A$  and  $B$ ,  $c(A) = c(B)$  if they are connected by a cable and  $|c(A) - c(B)| \leq 1$  otherwise. Also, find the number of cables used in the network.
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- 7 Eight problems were given to each of 30 students. After the test was given, point values of the problems were determined as follows: a problem is worth  $n$  points if it is not solved by exactly  $n$  contestants (no partial credit is given, only zero marks or full marks).
- (a) Is it possible that the contestant having got more points than any other contestant had also solved less problems than any other contestant?
- (b) Is it possible that the contestant having got less points than any other contestant has solved more problems than any other contestant?
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– **Geometry**

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- 1 A circle with center  $O$  is tangent to the sides of the angle with the vertex  $A$  at the points  $B$  and  $C$ . Let  $M$  be a point on the larger of the two arcs  $BC$  of this circle (different from  $B$  and  $C$ ) such that  $M$  does not lie on the line  $AO$ . Lines  $BM$  and  $CM$  intersect the line  $AO$  at the points  $P$  and  $Q$  respectively. Let  $K$  be the foot of the perpendicular drawn from  $P$  to  $AC$  and  $L$  be the foot of the perpendicular drawn from  $Q$  to  $AB$ . Prove that the lines  $OM$  and  $KL$  are perpendicular.
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- 2 Let  $I$  be the incenter of triangle  $ABC$ , and let  $A_1, B_1$ , and  $C_1$  be arbitrary points lying on segments  $AI, BI$ , and  $CI$ , respectively. The perpendicular bisectors of segments  $AA_1, BB_1$ , and  $CC_1$  form triangle  $A_2B_2C_2$ . Prove that the circumcenter of triangle  $A_2B_2C_2$  coincides with the circumcenter of triangle  $ABC$  if and only if  $I$  is the orthocenter of triangle  $A_1B_1C_1$ .
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- 3 Points  $M$  and  $M'$  are isogonal conjugates in the triangle  $ABC$ . We draw perpendiculars  $MP, MQ, MR$ , and  $M'P', M'Q', M'R'$  to the sides  $BC, AC, AB$  respectively. Let  $QR, Q'R'$ , and  $RP, R'P'$  and  $PQ, P'Q'$  intersect at  $E, F, G$  respectively. Show that the lines  $EA, FB$ , and  $GC$  are parallel.
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- 4 Let  $ABCD$  be a convex quadrilateral and let  $K, L, M, N$  be the midpoints of sides  $AB, BC, CD, DA$  respectively. Let  $NL$  and  $KM$  meet at point  $T$ . Show that  $8[DNTM] < [ABCD] < 8[DNTM]$ , where  $[P]$  denotes area of  $P$ .
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- 5 Let  $ABCD$  be a cyclic quadrilateral such that  $AB \cdot BC = 2 \cdot AD \cdot DC$ . Prove that its diagonals  $AC$  and  $BD$  satisfy the inequality  $8BD^2 \leq 9AC^2$ .

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6 Given a convex quadrilateral  $ABCD$ . The points  $P$  and  $Q$  are the midpoints of the diagonals  $AC$  and  $BD$  respectively. The line  $PQ$  intersects the lines  $AB$  and  $CD$  at  $N$  and  $M$  respectively. Prove that the circumcircles of triangles  $NAP$ ,  $NBQ$ ,  $MQD$ , and  $MPC$  have a common point.

7 Let  $ABCD$  be a cyclic quadrilateral whose interior angle at  $B$  is 60 degrees. Show that if  $BC = CD$ , then  $CD + DA = AB$ . Does the converse hold?

– **Number Theory**

1 Let  $n$  be a natural number and  $f_1, f_2, \dots, f_n$  be polynomials with integer coefficients. Show that there exists a polynomial  $g(x)$  which can be factored (with at least two terms of degree at least 1) over the integers such that  $f_i(x) + g(x)$  cannot be factored (with at least two terms of degree at least 1) over the integers for every  $i$ .

2 Let  $a, b, c$ , and  $d$  be positive integers satisfy the following properties:  
 (a) there are exactly 2004 pairs of real numbers  $(x, y)$  with  $0 \leq x, y \leq 1$  such that both  $ax + by$  and  $cx + dy$  are integers.  
 (b)  $\gcd(a, c) = 6$ .  
 Find  $\gcd(b, d)$ .

3 For any positive integer  $n$ , the sum  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$  is written in the lowest form  $\frac{p_n}{q_n}$ ; that is,  $p_n$  and  $q_n$  are relatively prime positive integers. Find all  $n$  such that  $p_n$  is divisible by 3.

4 Prove that there does not exist an integer  $n > 1$  such that  $n$  divides  $3^n - 2^n$ .

5 Find all integer solutions to  $y^2(x^2 + y^2 - 2xy - x - y) = (x + y)^2(x - y)$ .

6 Let  $p$  be a prime number, and let  $0 \leq a_1 < a_2 < \dots < a_m < p$  and  $0 \leq b_1 < b_2 < \dots < b_n < p$  be arbitrary integers. Denote by  $k$  the number of different remainders of  $a_i + b_j$ ,  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , modulo  $p$ . Prove that  
 (i) if  $m + n > p$ , then  $k = p$   
 (ii) if  $m + n \leq p$ , then  $k \geq m + n - 1$

7 Let  $A$  be a finite subset of prime numbers and  $a > 1$  be a positive integer. Show that the number of positive integers  $m$  for which all prime divisors of  $a^m - 1$  are in  $A$  is finite.