**Middle European Mathematical Olympiad 2009**

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by FelixD

**1** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(xf(y)) + f(f(x) + f(y)) = yf(x) + f(x + f(y))$$

holds for all $x$, $y \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

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**2** Suppose that we have $n \geq 3$ distinct colours. Let $f(n)$ be the greatest integer with the property that every side and every diagonal of a convex polygon with $f(n)$ vertices can be coloured with one of $n$ colours in the following way:

(i) At least two colours are used,

(ii) any three vertices of the polygon determine either three segments of the same colour or of three different colours.

Show that $f(n) \leq (n-1)^2$ with equality for infintely many values of $n$.

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**3** Let $ABCD$ be a convex quadrilateral such that $AB$ and $CD$ are not parallel and $AB = CD$. The midpoints of the diagonals $AC$ and $BD$ are $E$ and $F$, respectively. The line $EF$ meets segments $AB$ and $CD$ at $G$ and $H$, respectively. Show that $\angle AGH = \angle DHG$.

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**4** Determine all integers $k \geq 2$ such that for all pairs $(m, n)$ of different positive integers not greater than $k$, the number $n^{n-1} - m^{m-1}$ is not divisible by $k$.

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**5** Let $x$, $y$, $z$ be real numbers satisfying $x^2 + y^2 + z^2 + 9 = 4(x + y + z)$. Prove that

$$x^4 + y^4 + z^4 + 16(x^2 + y^2 + z^2) \geq 8(x^3 + y^3 + z^3) + 27$$

and determine when equality holds.

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**6** Let $a$, $b$, $c$ be real numbers such that for every two of the equations

$$x^2 + ax + b = 0, \quad x^2 + bx + c = 0, \quad x^2 + cx + a = 0$$

there is exactly one real number satisfying both of them. Determine all possible values of $a^2 + b^2 + c^2$.

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**7** The numbers $0, 1, \dots, n$ ($n \geq 2$) are written on a blackboard. In each step we erase an integer which is the arithmetic mean of two different numbers which are still left on the blackboard.

We make such steps until no further integer can be erased. Let $g(n)$ be the smallest possible number of integers left on the blackboard at the end. Find $g(n)$ for every $n$.

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**8** We colour every square of the $2009$ x $2009$ board with one of $n$ colours (we do not have to use every colour). A colour is called connected if either there is only one square of that colour or any two squares of the colour can be reached from one another by a sequence of moves of a chess queen without intermediate stops at squares having another colour (a chess quen moves horizontally, vertically or diagonally). Find the maximum $n$, such that for every colouring of the board at least on colour present at the board is connected.

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**9** Let $ABCD$ be a parallelogram with $\angle BAD = 60$ and denote by $E$ the intersection of its diagonals. The circumcircle of triangle $ACD$ meets the line $BA$ at $K \neq A$, the line $BD$ at $P \neq D$ and the line $BC$ at $L \neq C$. The line $EP$ intersects the circumcircle of triangle $CEL$ at points $E$ and $M$. Prove that triangles $KLM$ and $CAP$ are congruent.

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**10** Suppose that $ABCD$ is a cyclic quadrilateral and $CD = DA$. Points $E$ and $F$ belong to the segments $AB$ and $BC$ respectively, and $\angle ADC = 2\angle EDF$. Segments $DK$ and $DM$ are height and median of triangle $DEF$, respectively. $L$ is the point symmetric to $K$ with respect to $M$. Prove that the lines $DM$ and $BL$ are parallel.

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**11** Find all pairs $(m, n)$ of integers which satisfy the equation

$$(m+n)^4 = m^2n^2 + m^2 + n^2 + 6mn.$$

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**12** Find all non-negative integer solutions of the equation

$$2^x + 2009 = 3^y 5^z.$$

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