**Middle European Mathematical Olympiad 2010**

www.artofproblemsolving.com/community/c3583

by Martin N.

**1** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$, we have

$$f(x+y)+f(x)f(y)=f(xy)+(y+1)f(x)+(x+1)f(y).$$

---

**2** All positive divisors of a positive integer $N$ are written on a blackboard. Two players $A$ and $B$ play the following game taking alternate moves. In the firt move, the player $A$ erases $N$. If the last erased number is $d$, then the next player erases either a divisor of $d$ or a multiple of $d$. The player who cannot make a move loses. Determine all numbers $N$ for which $A$ can win independently of the moves of $B$.

*(4th Middle European Mathematical Olympiad, Individual Competition, Problem 2)*

---

**3** We are given a cyclic quadrilateral $ABCD$ with a point $E$ on the diagonal $AC$ such that $AD=AE$ and $CB=CE$. Let $M$ be the center of the circumcircle $k$ of the triangle $BDE$. The circle $k$ intersects the line $AC$ in the points $E$ and $F$. Prove that the lines $FM$, $AD$ and $BC$ meet at one point.

*(4th Middle European Mathematical Olympiad, Individual Competition, Problem 3)*

---

**4** Find all positive integers $n$ which satisfy the following tow conditions:
(a) $n$ has at least four different positive divisors;
(b) for any divisors $a$ and $b$ of $n$ satisfying $1<a<b<n$, the number $b-a$ divides $n$.

*(4th Middle European Mathematical Olympiad, Individual Competition, Problem 4)*

---

**5** Three strictly increasing sequences

$$a_1,a_2,a_3,\ldots,\qquad b_1,b_2,b_3,\ldots,\qquad c_1,c_2,c_3,\ldots$$

of positive integers are given. Every positive integer belongs to exactly one of the three sequences. For every positive integer $n$, the following conditions hold:
(a) $c_{a_n}=b_n+1$;
(b) $a_{n+1}>b_n$;
(c) the number $c_{n+1}c_n-(n+1)c_{n+1}-nc_n$ is even.
Find $a_{2010}$, $b_{2010}$ and $c_{2010}$.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 1)*

---

**6** For each integer $n \geqslant 2$, determine the largest real constant $C_n$ such that for all positive real numbers $a_1, \ldots, a_n$ we have

$$\frac{a_1^2+\ldots+a_n^2}{n} \geqslant \left(\frac{a_1+\ldots+a_n}{n}\right)^2 + C_n \cdot (a_1-a_n)^2.$$

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 2)*

---

**7** In each vertex of a regular $n$-gon, there is a fortress. At the same moment, each fortress shoots one of the two nearest fortresses and hits it. The *result of the shooting* is the set of the hit fortresses; we do not distinguish whether a fortress was hit once or twice. Let $P(n)$ be the number of possible results of the shooting. Prove that for every positive integer $k \geqslant 3$, $P(k)$ and $P(k+1)$ are relatively prime.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 3)*

---

**8** Let $n$ be a positive integer. A square $ABCD$ is partitioned into $n^2$ unit squares. Each of them is divided into two triangles by the diagonal parallel to $BD$. Some of the vertices of the unit squares are colored red in such a way that each of these $2n^2$ triangles contains at least one red vertex. Find the least number of red vertices.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 4)*

---

**9** The incircle of the triangle $ABC$ touches the sides $BC$, $CA$, and $AB$ in the points $D$, $E$ and $F$, respectively. Let $K$ be the point symmetric to $D$ with respect to the incenter. The lines $DE$ and $FK$ intersect at $S$. Prove that $AS$ is parallel to $BC$.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 5)*

---

**10** Let $A$, $B$, $C$, $D$, $E$ be points such that $ABCD$ is a cyclic quadrilateral and $ABDE$ is a parallelogram. The diagonals $AC$ and $BD$ intersect at $S$ and the rays $AB$ and $DC$ intersect at $F$. Prove that $\sphericalangle AFS = \sphericalangle ECD$.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 6)*

---

**11** For a nonnegative integer $n$, define $a_n$ to be the positive integer with decimal representation

$$1\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}1.$$

Prove that $\frac{a_n}{3}$ is always the sum of two positive perfect cubes but never the sum of two perfect squares.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 7)*

---

 

**12** We are given a positive integer $n$ which is not a power of two. Show that ther exists a positive integer $m$ with the following two properties:
(a) $m$ is the product of two consecutive positive integers;
(b) the decimal representation of $m$ consists of two identical blocks with $n$ digits.

*(4th Middle European Mathematical Olympiad, Team Competition, Problem 8)*

---