Middle European Mathematical Olympiad 2011

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– Individual Competition

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1 Initially, only the integer 44 is written on a board. An integer a on the board can be re- placed with four pairwise different integers a_1, a_2, a_3, a_4 such that the arithmetic mean $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$ of the four new integers is equal to the number a. In a step we simultaneously replace all the integers on the board in the above way. After 30 steps we end up with $n = 4^{30}$ integers $b_1, b2, \dots, b_n$ on the board. Prove that

$$\frac{b_1^2 + b_2^2 + b_3^2 + \dots + b_n^2}{n} \geq 2011.$$

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2 Let $n \geq 3$ be an integer. John and Mary play the following game: First John labels the sides of a regular n-gon with the numbers $1, 2, \dots, n$ in whatever order he wants, using each number exactly once. Then Mary divides this n-gon into triangles by drawing $n - 3$ diagonals which do not intersect each other inside the n-gon. All these diagonals are labeled with number 1. Into each of the triangles the product of the numbers on its sides is written. Let S be the sum of those $n - 2$ products.

Determine the value of S if Mary wants the number S to be as small as possible and John wants S to be as large as possible and if they both make the best possible choices.

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3 In a plane the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ with centers I_1 and I_2, respectively, intersect in two points A and B. Assume that $\angle I_1 A I_2$ is obtuse. The tangent to $\mathcal{K}_1$ in A intersects $\mathcal{K}_2$ again in C and the tangent to $\mathcal{K}_2$ in A intersects $\mathcal{K}_1$ again in D. Let $\mathcal{K}_3$ be the circumcircle of the triangle BCD. Let E be the midpoint of that arc CD of $\mathcal{K}_3$ that contains B. The lines AC and AD intersect $\mathcal{K}_3$ again in K and L, respectively. Prove that the line AE is perpendicular to KL.

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4 Let k and m, with $k > m$, be positive integers such that the number $km(k^2 - m^2)$ is divisible by $k^3 - m^3$. Prove that $(k - m)^3 > 3km$.

– Team Competition

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1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$y^2 f(x) + x^2 f(y) + xy = xyf(x + y) + x^2 + y^2$$

holds for all $x, y \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

2 Let a, b, c be positive real numbers such that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=2.$$

Prove that

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\geq\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}.$$

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3 For an integer $n\geq 3$, let $\mathcal{M}$ be the set $\{(x,y)|x,y\in\mathbb{Z},1\leq x\leq n,1\leq y\leq n\}$ of points in the plane.

What is the maximum possible number of points in a subset $S\subseteq\mathcal{M}$ which does not contain three distinct points being the vertices of a right triangle?

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4 Let $n\geq 3$ be an integer. At a MEMO-like competition, there are $3n$ participants, there are n languages spoken, and each participant speaks exactly three different languages. Prove that at least $\left\lceil\frac{2n}{9}\right\rceil$ of the spoken languages can be chosen in such a way that no participant speaks more than two of the chosen languages.

Note. $\lceil x\rceil$ is the smallest integer which is greater than or equal to x.

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5 Let $ABCDE$ be a convex pentagon with all five sides equal in length. The diagonals AD and EC meet in S with $\angle ASE=60^\circ$. Prove that $ABCDE$ has a pair of parallel sides.

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6 Let ABC be an acute triangle. Denote by B_0 and C_0 the feet of the altitudes from vertices B and C, respectively. Let X be a point inside the triangle ABC such that the line BX is tangent to the circumcircle of the triangle AXC_0 and the line CX is tangent to the circumcircle of the triangle AXB_0. Show that the line AX is perpendicular to BC.

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7 Let A and B be disjoint nonempty sets with $A\cup B=\{1,2,3,\ldots,10\}$. Show that there exist elements $a\in A$ and $b\in B$ such that the number $a^3+ab^2+b^3$ is divisible by 11.

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8 We call a positive integer n *amazing* if there exist positive integers a, b, c such that the equality

$$n=(b,c)(a,bc)+(c,a)(b,ca)+(a,b)(c,ab)$$

holds. Prove that there exist 2011 consecutive positive integers which are *amazing*.

Note. By (m,n) we denote the greatest common divisor of positive integers m and n.
