Middle European Mathematical Olympiad 2012

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by syk0526

– Individual Competition

– September 8th

1 Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $\mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x+f(y)) = yf(xy+1)$$

holds for all $x, y \in \mathbb{R}^+$.

2 Let N be a positive integer. A set $S \subset \{1, 2, \cdots, N\}$ is called *allowed* if it does not contain three distinct elements a, b, c such that a divides b and b divides c. Determine the largest possible number of elements in an allowed set S.

3 In a given trapezium $ABCD$ with AB parallel to CD and $AB > CD$, the line BD bisects the angle $\angle ADC$. The line through C parallel to AD meets the segments BD and AB in E and F, respectively. Let O be the circumcenter of the triangle BEF. Suppose that $\angle ACO = 60^\circ$. Prove the equality

$$CF = AF + FO.$$

4 The sequence $\{a_n\}_{n\geq 0}$ is defined by $a_0 = 2, a_1 = 4$ and

$$a_{n+1} = \frac{a_n a_{n-1}}{2} + a_n + a_{n-1}$$

for all positive integers n. Determine all prime numbers p for which there exists a positive integer m such that p divides the number $a_m - 1$.

– Team Competition

– October 9th

1 Find all triplets (x, y, z) of real numbers such that

$$2x^3 + 1 = 3zx$$
$$2y^3 + 1 = 3xy$$
$$2z^3 + 1 = 3yz$$

2 Let a, b and c be positive real numbers with $abc = 1$. Prove that

$$\sqrt{9+16a^2}+\sqrt{9+16b^2}+\sqrt{9+16c^2} \geq 3+4(a+b+c)$$

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3 Let n be a positive integer. Consider words of length n composed of letters from the set $\{M, E, O\}$. Let a be the number of such words containing an even number (possibly 0) of blocks ME and an even number (possibly 0) blocks of MO . Similarly let b the number of such words containing an odd number of blocks ME and an odd number of blocks MO. Prove that $a > b$.

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4 Let $p > 2$ be a prime number. For any permutation $\pi = (\pi(1), \pi(2), \cdots, \pi(p))$ of the set $S = \{1, 2, \cdots, p\}$, let $f(\pi)$ denote the number of multiples of p among the following p numbers:

$$\pi(1), \pi(1)+\pi(2), \cdots, \pi(1)+\pi(2)+\cdots+\pi(p)$$

Determine the average value of $f(\pi)$ taken over all permutations π of S.

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5 Let K be the midpoint of the side AB of a given triangle ABC. Let L and M be points on the sides AC and BC, respectively, such that $\angle CLK = \angle KMC$. Prove that the perpendiculars to the sides AB, AC, and BC passing through K, L, and M, respectively, are concurrent.

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6 Let $ABCD$ be a convex quadrilateral with no pair of parallel sides, such that $\angle ABC = \angle CDA$. Assume that the intersections of the pairs of neighbouring angle bisectors of $ABCD$ form a convex quadrilateral $EFGH$. Let K be the intersection of the diagonals of $EFGH$. Prove that the lines AB and CD intersect on the circumcircle of the triangle BKD.

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7 Find all triplets (x, y, z) of positive integers such that

$$x^y + y^x = z^y$$

$$x^y + 2012 = y^{z+1}$$

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8 For any positive integer n let $d(n)$ denote the number of positive divisors of n. Do there exist positive integers a and b, such that $d(a) = d(b)$ and $d(a^2) = d(b^2)$, but $d(a^3) \neq d(b^3)$?
