Middle European Mathematical Olympiad 2013

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– Individual Competition

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1 Let a, b, c be positive real numbers such that

$$a+b+c=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Prove that

$$2(a+b+c) \geq \sqrt[3]{7a^2b+1}+\sqrt[3]{7b^2c+1}+\sqrt[3]{7c^2a+1}.$$

Find all triples (a, b, c) for which equality holds.

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2 Let n be a positive integer. On a board consisting of $4n \times 4n$ squares, exactly $4n$ tokens are placed so that each row and each column contains one token. In a step, a token is moved horizontally of vertically to a neighbouring square. Several tokens may occupy the same square at the same time. The tokens are to be moved to occupy all the squares of one of the two diagonals. Determine the smallest number $k(n)$ such that for any initial situation, we can do it in at most $k(n)$ steps.

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3 Let ABC be an isosceles triangle with $AC = BC$. Let N be a point inside the triangle such that $2\angle ANB = 180^\circ + \angle ACB$. Let D be the intersection of the line BN and the line parallel to AN that passes through C. Let P be the intersection of the angle bisectors of the angles CAN and ABN. Show that the lines DP and AN are perpendicular.

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4 Let a and b be positive integers. Prove that there exist positive integers x and y such that

$$\binom{x+y}{2}=ax+by.$$

– Team Competition

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1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x)+2y)=f(x^2)+f(y)+x+y-1$$

holds for all $x, y \in \mathbb{R}$.

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2 Let x, y, z, w be nonzero real numbers such that $x + y \neq 0$, $z + w \neq 0$, and $xy + zw \geq 0$. Prove that

$$\left(\frac{x+y}{z+w}+\frac{z+w}{x+y}\right)^{-1}+\frac{1}{2}\geq\left(\frac{x}{z}+\frac{z}{x}\right)^{-1}+\left(\frac{y}{w}+\frac{w}{y}\right)^{-1}$$

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3 There are $n \geq 2$ houses on the northern side of a street. Going from the west to the east, the houses are numbered from 1 to n. The number of each house is shown on a plate. One day the inhabitants of the street make fun of the postman by shuffling their number plates in the following way: for each pair of neighbouring houses, the currnet number plates are swapped exactly once during the day.
How many different sequences of number plates are possible at the end of the day?

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4 Consider finitely many points in the plane with no three points on a line. All these points can be coloured red or green such that any triangle with vertices of the same colour contains at least one point of the other colour in its interior.
What is the maximal possible number of points with this property?

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5 Let ABC be and acute triangle. Construct a triangle PQR such that $AB = 2PQ$, $BC = 2QR$, $CA = 2RP$, and the lines PQ, QR, and RP pass through the points A, B, and C, respectively. (All six points A, B, C, P, Q, and R are distinct.)

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6 Let K be a point inside an acute triangle ABC, such that BC is a common tangent of the circumcircles of AKB and AKC. Let D be the intersection of the lines CK and AB, and let E be the intersection of the lines BK and AC . Let F be the intersection of the line BC and the perpendicular bisector of the segment DE. The circumcircle of ABC and the circle k with centre F and radius FD intersect at points P and Q.
Prove that the segment PQ is a diameter of k.

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7 The numbers from 1 to 2013^2 are written row by row into a table consisting of 2013×2013 cells. Afterwards, all columns and all rows containing at least one of the perfect squares $1, 4, 9, \cdots, 2013^2$ are simultaneously deleted.
How many cells remain?

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8 The expression

$$\pm\square\pm\square\pm\square\pm\square\pm\square\pm\square$$

is written on the blackboard. Two players, A and B, play a game, taking turns. Player A takes the first turn. In each turn, the player on turn replaces a symbol $\square$ by a positive integer. After all the symbols $\square$ are replace, player A replaces each of the signs $\pm$ by either + or -, independently of each other. Player A wins if the value of the expression on the blackboard is not divisible by any of the numbers $11, 12, \cdots, 18$. Otherwise, player B wins. Determine which player has a winning strategy.
