Middle European Mathematical Olympiad 2014

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–	Individual Competition
1	Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $$xf(y)+f(xf(y))-xf(f(y))-f(xy)=2x+f(y)-f(x+y)$$ holds for all $x,y\in\mathbb{R}$.
2	We consider dissections of regular n-gons into $n-2$ triangles by $n-3$ diagonals which do not intersect inside the n-gon. A *bicoloured triangulation* is such a dissection of an n-gon in which each triangle is coloured black or white and any two triangles which share an edge have different colours. We call a positive integer $n\geq 4$ *triangulable* if every regular n-gon has a bicoloured triangulation such that for each vertex A of the n-gon the number of black triangles of which A is a vertex is greater than the number of white triangles of which A is a vertex. Find all triangulable numbers.
3	Let ABC be a triangle with $AB<AC$ and incentre I. Let E be the point on the side AC such that $AE=AB$. Let G be the point on the line EI such that $\angle IBG=\angle CBA$ and such that E and G lie on opposite sides of I. Prove that the line AI, the line perpendicular to AE at E, and the bisector of the angle $\angle BGI$ are concurrent.
4	For integers $n\geq k\geq 0$ we define the *bibinomial coefficient* $\left(\binom{n}{k}\right)$ by $$\left(\binom{n}{k}\right)=\frac{n!!}{k!!(n-k)!!}.$$ Determine all pairs (n,k) of integers with $n\geq k\geq 0$ such that the corresponding bibinomial coefficient is an integer. [i]Remark: The double factorial $n!!$ is defined to be the product of all even positive integers up to n if n is even and the product of all odd positive integers up to n if n is odd. So e.g. $0!!=1$, $4!!=2\cdot 4=8$, and $7!!=1\cdot 3\cdot 5\cdot 7=105$.[/i]
–	Team Competition

1 Determine the lowest possible value of the expression

$$\frac{1}{a+x}+\frac{1}{a+y}+\frac{1}{b+x}+\frac{1}{b+y}$$

where $a, b, x,$ and y are positive real numbers satisfying the inequalities

$$\frac{1}{a+x} \geq \frac{1}{2}$$
$$\frac{1}{a+y} \geq \frac{1}{2}$$
$$\frac{1}{b+x} \geq \frac{1}{2}$$
$$\frac{1}{b+y} \geq 1.$$

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2 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(xy) + xyf(x) \geq f(x^2)f(y) + x^2y$$

holds for all $x, y \in \mathbb{R}$.

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3 Let K and L be positive integers. On a board consisting of $2K \times 2L$ unit squares an ant starts in the lower left corner square and walks to the upper right corner square. In each step it goes horizontally or vertically to a neighbouring square. It never visits a square twice. At the end some squares may remain unvisited.
In some cases the collection of all unvisited squares forms a single rectangle. In such cases, we call this rectangle *MEMOrable*.

Determine the number of different MEMOrable rectangles.

Remark: Rectangles are different unless they consist of exactly the same squares.

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4 In Happy City there are 2014 citizens called $A_1, A_2, \dots, A_{2014}$. Each of them is either *happy* or *unhappy* at any moment in time. The mood of any citizen A changes (from being unhappy to being happy or vice versa) if and only if some other happy citizen smiles at A. On Monday morning there were N happy citizens in the city.

The following happened on Monday during the day: the citizen A_1 smiled at citizen A_2, then A_2 smiled at A_3, etc., and, finally, A_{2013} smiled at A_{2014}. Nobody smiled at anyone else apart from this. Exactly the same repeated on Tuesday, Wednesday and Thursday. There were exactly 2000 happy citizens on Thursday evening.

Determine the largest possible value of N.

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5 Let ABC be a triangle with $AB < AC$. Its incircle with centre I touches the sides BC, CA, and AB in the points D, E, and F respectively. The angle bisector AI intersects the lines DE and DF in the points X and Y respectively. Let Z be the foot of the altitude through A with respect to BC.

Prove that D is the incentre of the triangle XYZ.

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6 Let the incircle k of the triangle ABC touch its side BC at D. Let the line AD intersect k at $L \neq D$ and denote the excentre of ABC opposite to A by K. Let M and N be the midpoints of BC and KM respectively.

Prove that the points B, C, N, and L are concyclic.

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7 A finite set of positive integers A is called *meanly* if for each of its nonempy subsets the arithmetic mean of its elements is also a positive integer. In other words, A is meanly if $\frac{1}{k}(a_1 + \cdots + a_k)$ is an integer whenever $k \geq 1$ and $a_1, \ldots, a_k \in A$ are distinct.

Given a positive integer n, determine the least possible sum of the elements of a meanly n-element set.

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8 Determine all quadruples (x, y, z, t) of positive integers such that

$$20^x + 14^{2y} = (x + 2y + z)^{zt}.$$
