**Romanian Masters In Mathematics 2012**

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by jgnr, WakeUp

**Day 1**

**1** Given a finite number of boys and girls, a *sociable set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *sociable set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

*(Poland) Marek Cygan*

**2** Given a non-isosceles triangle $ABC$, let $D, E$, and $F$ denote the midpoints of the sides $BC, CA$, and $AB$ respectively. The circle $BCF$ and the line $BE$ meet again at $P$, and the circle $ABE$ and the line $AD$ meet again at $Q$. Finally, the lines $DP$ and $FQ$ meet at $R$. Prove that the centroid $G$ of the triangle $ABC$ lies on the circle $PQR$.

*(United Kingdom) David Monk*

**3** Each positive integer is coloured red or blue. A function $f$ from the set of positive integers to itself has the following two properties:

(a) if $x \leq y$, then $f(x) \leq f(y)$; and
(b) if $x, y$ and $z$ are (not necessarily distinct) positive integers of the same colour and $x+y=z$, then $f(x)+f(y)=f(z)$.

Prove that there exists a positive number $a$ such that $f(x) \leq ax$ for all positive integers $x$.

*(United Kingdom) Ben Elliott*

**Day 2**

**4** Prove that there are infinitely many positive integers $n$ such that $2^{2^n+1}+1$ is divisible by $n$ but $2^n+1$ is not.

*(Russia) Valery Senderov*

**5** Given a positive integer $n \geq 3$, colour each cell of an $n \times n$ square array with one of $\lfloor (n+2)^2/3 \rfloor$ colours, each colour being used at least once. Prove that there is some $1 \times 3$ or $3 \times 1$ rectangular subarray whose three cells are coloured with three different colours.

*(Russia) Ilya Bogdanov, Grigory Chelnokov, Dmitry Khramtsov*

**6** Let $ABC$ be a triangle and let $I$ and $O$ denote its incentre and circumcentre respectively. Let $\omega_A$ be the circle through $B$ and $C$ which is tangent to the incircle of the triangle $ABC$; the circles $\omega_B$ and $\omega_C$ are defined similarly. The circles $\omega_B$ and $\omega_C$ meet at a point $A'$ distinct from $A$; the points $B'$ and $C'$ are defined similarly. Prove that the lines $AA', BB'$ and $CC'$ are concurrent at a point on the line $IO$.

*(Russia) Fedor Ivlev*