Harvard-MIT Mathematics Tournament 1998

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– Advanced Topics

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1 Evaluate $\sin(1998^\circ + 237^\circ)\sin(1998^\circ - 1653^\circ)$.

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2 How many values of x, $-19 < x < 98$, satisfy $\cos^2 x + 2\sin^2 x = 1$?

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3 Finds the sum of the infinite series $1 + 2\left(\frac{1}{1998}\right) + 3\left(\frac{1}{1998}\right)^2 + 4\left(\frac{1}{1998}\right)^3 + \cdots$.

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4 Find the range of $f(A) = \frac{\sin A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A\cos^2 A)}{\tan A(\sec A - \sin A\tan A)}$ if $A \neq \dfrac{n\pi}{2}$.

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5 How many positive integers less than 1998 are relatively prime to 1547? (Two integers are relatively prime if they have no common factors besides 1.)

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6 In the diagram below, how many distinct paths are there from January 1 to December 31, moving from one adjacent dot to the next either to the right, down, or diagonally down to the right?

$$\begin{array}{rcccccccccl}
\text{Jan. }1\!\rightarrow & * & * & * & * & * & * & * & * & * & * \\
 & * & * & & * & * & & * & * & & * \\
 & * & * & * & * & * & * & * & * & * & * \\
 & * & & & * & & & * & * & & * \\
 & * & * & * & * & * & * & * & * & * & * \leftarrow\!\text{Dec. }31
\end{array}$$

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7 The Houson Association of Mathematics Educators decides to hold a grand forum on mathematics education and invites a number of politicians from around the United States to participate. Around lunch time the politicians decide to play a game. In this game, players can score 19 points for pegging the coordinator of the gathering with a spit ball, 9 points for downing an entire cup of the forums interpretation of coffee, or 8 points for quoting more than three consecutive words from the speech Senator Bobbo delivered before lunch. What is the product of the two greatest scores that a player cannot score in this game?

8 Given any two positive real numbers x and y, then $x\Diamond y$ is a positive real number defined in terms of x and y by some fixed rule. Suppose the operation $x\Diamond y$ satisfies the equations $(x\cdot y)\Diamond y = x(y\Diamond y)$ and $(x\Diamond 1)\Diamond x = x\Diamond 1$ for all $x, y > 0$. Given that $1\Diamond 1 = 1$, find $19\Diamond 98$.

9 Bobs Rice ID number has six digits, each a number from 1 to 9, and any digit can be used any number of times. The ID number satifies the following property: the first two digits is a number divisible by 2, the first three digits is a number divisible by 3, etc. so that the ID number itself is divisible by 6. One ID number that satisfies this condition is 123252. How many different possibilities are there for Bobs ID number?

10 In the fourth annual Swirled Series, the Oakland Alphas are playing the San Francisco Gammas. The first game is played in San Francisco and succeeding games alternate in location. San Francisco has a 50% chance of winning their home games, while Oakland has a probability of 60% of winning at home. Normally, the series will stretch on forever until one team gets a three game lead, in which case they are declared the winners. However, after each game in San Francisco there is a 50% chance of an earthquake, which will cause the series to end with the team that has won more games declared the winner. What is the probability that the Gammas will win?

– Algebra

1 The cost of 3 hamburgers, 5 milk shakes, and 1 order of fries at a certain fast food restaurant is $\$23.50$. At the same restaurant, the cost of 5 hamburgers, 9 milk shakes, and 1 order of fries is $\$39.50$. What is the cost of 2 hamburgers, 2 milk shakes and 2 orders of fries at this restaurant?

2 Bobbo starts swimming at 2 feet/s across a 100 foot wide river with a current of 5 feet/s. Bobbo doesnt know that there is a waterfall 175 feet from where he entered the river. He realizes his predicament midway across the river. What is the minimum speed that Bobbo must increase to make it to the other side of the river safely?

3 Find the sum of all even positive integers less than 233 not divisible by 10.

4 Given that r and s are relatively prime positive integers such that $\frac{r}{s} = \frac{2(\sqrt{2}+\sqrt{10})}{5\left(\sqrt{3+\sqrt{5}}\right)}$, find r and s.

5 A man named Juan has three rectangular solids, each having volume 128. Two of the faces of one solid have areas 4 and 32. Two faces of another solid have areas 64 and 16. Finally, two faces of the last solid have areas 8 and 32. What is the minimum possible exposed surface area of the tallest tower Juan can construct by stacking his solids one on top of the other, face to face? (Assume that the base of the tower is not exposed.)

6 How many pairs of positive integers (a, b) with $a \le b$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$?

7 Given that three roots of $f(x) = x^4 + ax^2 + bx + c$ are 2, -3, and 5, what is the value of $a + b + c$?

8 Find the set of solutions for x in the inequality $\frac{x+1}{x+2} > \frac{3x+4}{2x+9}$ when $x \neq 2$, $x \neq -\frac{9}{2}$.

9 Suppose $f(x)$ is a rational function such that $3f\left(\frac{1}{x}\right) + \frac{2f(x)}{x} = x^2$ for $x \neq 0$. Find $f(-2)$.

10 G. H. Hardy once went to visit Srinivasa Ramanujan in the hospital, and he started the conversation with: I came here in taxi-cab number 1729. That number seems dull to me, which I hope isnt a bad omen. Nonsense, said Ramanujan. The number isnt dull at all. Its quite interesting. Its the smallest number that can be expressed as the sum of two cubes in two different ways. Ramujan had immediately seen that $1729 = 12^3 + 1^3 = 10^3 + 9^3$. What is the smallest positive integer representable as the sum of the cubes of *three* positive integers in two different ways?

– Calculus

1 Farmer Tim is lost in the densely-forested Cartesian plane. Starting from the origin he walks a sinusoidal path in search of home; that is, after t minutes he is at position $(t, \sin t)$.

Five minutes after he sets out, Alex enters the forest at the origin and sets out in search of Tim. He walks in such a way that after he has been in the forest for m minutes, his position is $(m, \cos t)$.

What is the greatest distance between Alex and Farmer Tim while they are walking in these paths?

2 A cube with sides 1m in length is filled with water, and has a tiny hole through which the water drains into a cylinder of radius 1 m. If the water level in the cube is falling at a rate of 1 cm/s, at what rate is the water level in the cylinder rising?

3 Find the area of the region bounded by the graphs $y = x^2$, $y = x$, and $x = 2$.

4 Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots$, for $-1 \le x \le 1$. Find $\sqrt{e^{\int_0^1 f(x)dx}}$.

5 Evaluate $\lim_{x \to 1} x^{\frac{x}{\sin(1-x)}}$.

6 Edward, the author of this test, had to escape from prison to work in the grading room today. He stopped to rest at a place $1,875$ feet from the prison and was spotted by a guard with a crossbow.

The guard fired an arrow with an initial velocity of $100\frac{\text{ft}}{\text{s}}$. At the same time, Edward started running away with an acceleration of $1\frac{\text{ft}}{\text{s}^2}$. Assuming that air resistance causes the arrow to decelerate at $1\frac{\text{ft}}{\text{s}^2}$, and that it does hit Edward, how fast was the arrow moving at the moment of impact (in $\frac{\text{ft}}{\text{s}}$)?

7 A parabola is inscribed in equilateral triangle ABC of side length 1 in the sense that AC and BC are tangent to the parabola at A and B, respectively.

Find the area between AB and the parabola.

8 Find the slopes of all lines passing through the origin and tangent to the curve $y^2 = x^3 + 39x - 35$.

9 Evaluate $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$.

10 Let S be the locus of all points (x, y) in the first quadrant such that $\frac{x}{t} + \frac{y}{1-t} = 1$ for some t with $0 < t < 1$. Find the area of S.

– Geometry

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